ON THE ALMOST AXISYMMETRIC FLOWS WITH FORCING TERMS.

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ON THE ALMOST AXISYMMETRIC FLOWS WITH FORCING TERMS.

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To my Parents with my deepest affection.

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SUMMARY

This work is concerned with the Almost Axisymmetric Flows with Forcing Terms which are derived from the inviscid Boussinesq equations. It is our hope that these flows will be useful in Meteorology to describe tropical cyclones. We show that these flows give rise to a collection of Monge-Ampere equations for which we prove an existence and uniqueness result. What makes these equations unusual is the boundary conditions they are expected to satisfy and the fact that the boundary is part of the unknown. Our study allows us to make inferences in a toy Almost Axisymmetric Flows with a forcing term model.

CHAPTER I

INTRODUCTION

In this work, we consider the so-called Almost Axisymmetric Flows with Forcing terms in the absence of viscosity. The variant of these flows we are concerned with, originated in a work by Craig [17] and was built on works by several atmospheric scientists (e.g., [23] [28] [43]). The Almost Axisymmetric Flows are designed to study the structure of tropical cyclones and were suspected to possess a Hamiltonian structure. As we will show later, the equations describing the Almost Axisymmetric Flows with Forcing Terms are derived as an approximation to the inviscid Boussinesq equations.

This work focuses on the free boundary version obtained by Cullen, following a procedure proposed by Craig [17] and Shutts [43]. In the cylindrical polar coordinates (λ, r, z), the time dependent domain where the fluid evolves is of the form

$$\Gamma_{\varsigma^t} = \{(\lambda, r, z) : 0 \leq \lambda \leq 2\pi, \ \ 0 \leq z \leq H, \ \ r_0 \leq r \leq \varsigma(t, \lambda, z)\} \tag{1.0.1}$$

where the boundary $r = \varsigma^t$ is a material surface and r_0, H are positif real numbers. We have used the notation $S^t = S(t, \cdot, \cdot)$. The temperature $\theta'(t, \lambda, r, z)$ within the domain of the vortex (where the PDEs are considered) is higher than the temperature in the ambient fluid which is maintained at a constant temperature θ_0 in a rotating framework where the coriolis coefficient is Ω. We denote by $\mathbf{u} = (u, v, w)$ the velocity the fluid in cylindrical coordinates. The material derivative associated to this velocity in cylindrical coordinates takes the form $\frac{D}{Dt} := \frac{\partial}{\partial t} + \frac{u}{r}\frac{\partial}{\partial \lambda} + v\frac{\partial}{\partial t} + w\frac{\partial}{\partial z}$. The pressure inside the vortex is denoted by φ.

The unknown of the problem are $\mathbf{u} = (u, v, w), \theta', \varphi, \varsigma$ while the equations describing the Almost Axisymmetric Flows with Forcing Terms are given by

$$\begin{cases} \frac{\mathrm{D}u}{\mathrm{D}t} + \frac{uv}{r} + 2\Omega v + \frac{1}{r}\frac{\partial \varphi}{\partial \lambda} & = \frac{1}{r}F(t,\lambda,r,z), & (1.0.2a) \\ \frac{\mathrm{D}\theta'}{\mathrm{D}t} & = S(t,\lambda,r,z), & (1.0.2b) \\ \frac{u^2}{r} + 2\Omega u & = \frac{\partial \varphi}{\partial r}, & (1.0.2c) \\ \frac{\partial \varphi}{\partial z} - g\frac{\theta'}{\theta_0} & = 0 & (1.0.2d) \\ \frac{1}{r}\frac{\partial}{\partial \lambda}(u) + \frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{\partial w}{\partial z} & = 0, & (1.0.2e) \end{cases}$$

The conditions on the boundary are given by

$$\begin{cases} \mathbf{u_t}\cdot\mathbf{n_t} = 0 & \text{on } \{r = r_0\}\cup\{z=0\}\cup\{z=H\} \\ \frac{\partial \varsigma_t}{\partial t} + \frac{u}{r}\frac{\partial \varsigma_t}{\partial t} + w\frac{\partial \varsigma_t}{\partial z} = v & \textit{on } \{r = \varsigma(t,\lambda,z)\} \end{cases} \tag{1.0.3}$$

along with the condition

$$\varphi(t,\lambda,\varsigma(t,\lambda,H),H) = 0.$$

Here $\mathbf{n_t}$ is the unit outward normal vector field at time t and

$F(t,\lambda,r,z)$ and $S(t,\lambda,r,z)$ are prescribed forcing terms of the system. These equations are supplemented by the initial conditions

$$\mathbf{u}_{|t=0} = \mathbf{u}_0 = (u_0, v_0, w_0);\ \theta'_{|t=0} = \theta'_0\ \ \varphi_{|t=0} = \varphi_0$$

The solutions we are interested in are the ones that are stable in some sense to be made precised later. We require initially the following conditions :

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial \lambda}(u_0) + \frac{1}{r}\frac{\partial}{\partial r}(rv_0) + \frac{\partial w_0}{\partial z} = 0 \\ \nabla^2_{r,z}\left(\varphi_0 + \Omega^2\frac{r^2}{2}\right) > 0 \\ \frac{u_0^2}{r} + 2\Omega u_0 = \frac{\partial \varphi_0}{\partial r} \\ \frac{\partial \varphi_0}{\partial z} - g\frac{\theta'_0}{\theta_0} = 0 \end{cases} \tag{1.0.4}$$

1.1 *Derivation of the almost axisymmetric flow equations with forcing terms from Boussinesq equations.*

The equations in (1.0.2) are obtained as approximations to the well known 3D inviscid Boussinesq equations with forcing terms:

$$\begin{cases} \frac{\bar{D}\bar{\mathbf{u}}}{Dt} + 2\Omega\bar{\mathbf{u}} \times \partial_x + \nabla\bar{\varphi} - \frac{g}{\theta_0}\bar{\theta}e_3 - \bar{\mathbf{F}} = \mathbf{0} \\ \frac{\bar{D}\bar{\theta}}{Dt} = \bar{S} \\ \mathrm{div}(\bar{\mathbf{u}}) = 0 \end{cases} \tag{1.1.1}$$

Here, $\frac{\bar{D}}{Dt} := \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial\lambda} + \bar{v}\frac{\partial}{\partial t} + \bar{w}\frac{\partial}{\partial z}$. These equations are written in the cartesian coordinates (x, y, z) and the velocity field $\bar{\mathbf{u}}$ is expressed in the associated basis $(\partial_x, \partial_y, \partial_z)$. Here $\bar{\varphi}$ is the pressure, $\bar{\theta}$ is the temperature, $\bar{\mathbf{F}}, \bar{S}$ are respectively a vector and a scalar forcing terms. As we show in section 2 of Chapter 2, under the cylindrical coordinate change of variables, the equations in (1.1.1) transform into

$$\begin{cases} \dfrac{\mathrm{D}u}{\mathrm{D}t} + \dfrac{uv}{r} + 2\Omega v + \dfrac{1}{r}\dfrac{\partial\varphi}{\partial\lambda} &= \frac{1}{r}F(t,\lambda,r,z), & (1.1.2\text{a}) \\ \dfrac{\mathrm{D}\theta'}{\mathrm{D}t} &= S(t,\lambda,r,z), & (1.1.2\text{b}) \\ \dfrac{\mathrm{D}v}{\mathrm{D}t} + \dfrac{u^2}{r} + 2\Omega u &= \frac{\partial\varphi}{\partial r}, & (1.1.2\text{c}) \\ \dfrac{\mathrm{D}w}{\mathrm{D}t} + \dfrac{\partial\varphi}{\partial z} - g\dfrac{\theta'}{\theta_0} &= 0 & (1.1.2\text{d}) \\ \dfrac{1}{r}\dfrac{\partial}{\partial\lambda}(u) + \dfrac{1}{r}\dfrac{\partial}{\partial r}(rv) + \dfrac{\partial w}{\partial z} &= 0, & (1.1.2\text{e}) \end{cases}$$

Boussinesq equations are known to model large scale flows in the atmosphere where the vertical component of the velocity field is much smaller than the horizontal component. This leads to the hydrostatic approximation where $\frac{Dw}{Dt} \approx 0$. In addition, in a rotating setting, the radial velocity of fluid is assumed to be small. To take this fact into account in our equations, we also make the approximation $\frac{Dv}{Dt} \approx 0$. Remarkably enough, it was predicted by Craig [17] that these approximations do not destroy the Hamiltonian structure of the Almost Axisymmetric Flows. Thus, from (1.1.2) we obtain (1.0.2).

1.2 Change of variables into the Dual space and formal Justification.

The computations performed in this subsection are valid if we are dealing with functions which are smooth enough.

Our approach to the problem (1.0.2) relies on a specific change of variables involving the expressions $ur + \Omega r^2$ and $\frac{g}{\theta_0}\theta'$ into a space we called "the dual space." As we will soon see, this appropriately chosen change of variables is motivated by interesting behaviors of the balanced axisymmetric vortex. Indeed, In the context of axisymmetric flows, namely when the quantities involved in equations (1.0.2) are all independent of λ and $F = S = 0$, the angular momentum $ur + \Omega r^2$ and the potential temperature θ' turn out to be key conserved quantities along trajectories of the flow :

$$\frac{D}{Dt}(ur + \Omega r^2) = \frac{D}{Dt}(\frac{g}{\theta_0}\theta') = 0$$

Moreover, the same quantities somehow control the rate of change of some variant of the pressure φ in the radial and vertical direction in the following way:

$$(ur + \Omega r^2)^2 = r^3 \frac{\partial}{\partial r}[\varphi + \frac{\Omega^2}{2} r^2]; \qquad \frac{g}{\theta_0}\theta = \frac{\partial}{\partial z}[\varphi + \frac{\Omega^2}{2} r^2] \tag{1.2.1}$$

If we introduce new variables $P = \varphi + \frac{\Omega^2}{2} r^2$ and $2s = r_0^{-2} - r^{-2}$ then (1.2.1) becomes

$$\left(ur + \Omega r^2\right)^2 = \partial_s P, \qquad \frac{g}{\theta_0}\theta' = \partial_z P$$

These considerations suggest to study the system in new coordinates λ, $\Upsilon = \partial_s P$ and $Z = \partial_s P$. The new coordinates have the principal advantage of shedding light on the strucure of the Almost Axisymmetric Flows equations and making more transparent the main mechanisms driving the system. More specifically, in the new variables, the Almost Axisymmetric Flows with Forcing terms have a very simple formulation in terms of the continuity equation

$$\begin{cases} \frac{\partial \sigma}{\partial t} + div(\sigma \mathbf{X}_\sigma) = 0 \\ \sigma_{|t=0} = \sigma_0 \end{cases} \tag{1.2.2}$$

Here,

$$\mathbf{X}_\sigma =$$
$$\left(\frac{\sqrt{\Upsilon}}{r_0^2} - \Omega - 2\sqrt{\Upsilon}\frac{\partial \Psi}{\partial \Upsilon},\ 2\sqrt{\Upsilon}\left[F_t\left(\lambda, \frac{r_0}{\sqrt{1-2r_0^2 \frac{\partial \Psi}{\partial \Upsilon}}}, \frac{\partial \Psi}{\partial Z}\right) + \frac{\partial \Psi}{\partial \lambda}\right],\ \frac{g}{\theta_0} S_t\left(\lambda, \frac{r_0}{\sqrt{1-2r_0^2 \frac{\partial \Psi}{\partial \Upsilon}}}, \frac{\partial \Psi}{\partial Z}\right)\right) \tag{1.2.3}$$

and there exists P such that P_λ and Ψ_λ are Legendre transforms of each other and solve the Monge-Ampere equation (1.4.1).

1.3 Comparisons with other studies.

The other well known PDE whose study comes close to the inviscid Boussinesq equations is the incompressible Euler equation. Recently, the free boundary Euler equations have been studied intensively by different groups of people including Linblad [38] Shatah-Zeng [44], Coutand-Shkoller [12]. For the well-posedness of the free boundary problem in incompressible Euler equations, it is required that the pressure satisfies the following condition:

$$\nabla \varphi_t \cdot \mathbf{n}_t < c_0 < 0 \qquad \text{on the free boundary}$$

The above condition was instrumental to obtain an existence and well-posedness result. This suggests that in the case of the Almost Axisymmetric symmetric Flows a certain condition is expected to be imposed on the pressure. From a physical perspective, we are interested in solutions that are stable in the sense that they correspond to a minimum energy state with respect to parcel displacements that preserve the angular momentum and the potential temperature (see [23]). For a solution to be stable we require that

$$\nabla^2_{r,z}\left(\frac{\Omega^2 r^2}{2} + \varphi_t^\lambda(r,z)\right) > 0 \tag{1.3.1}$$

Note that this conditon implies that $\frac{\Omega^2 r^2}{2} + \varphi_t^\lambda(r,z)$ is strictly convex and validates the change of variables discussed in the previous section.

1.4 Hamiltonian and a Monge-Ampere equation.

The Almost Axisymmtric Flows with Forcing Term comes along with an energy functional which proves to be an important tool in the solution procedure that we propose. When these flows evolve with a velocity $\mathbf{u} = (u, v, w)$ and temperature θ', the density energy is given by

$$\frac{u^2}{2} + \frac{g}{\theta_0}\theta'$$

In the variables (s, Υ, Z) that we introduced in section 1.2, this energy density takes the form

$$\frac{u^2}{2} + \frac{g}{\theta_0}\theta' = -s\Upsilon - zZ + \frac{\Upsilon}{2r_0^2} + \Omega\sqrt{\Upsilon} + \frac{\Omega r^2}{2(1-2r_0^2 s)}$$

The measure in the physical space can be expressed in different variables. Computing the Jacobian of the change of variables, one shows the existence of a scalar function σ such that

$$\chi_{\Gamma_\varsigma}(\lambda, r, z) r d\lambda dr dz = \chi_{D_h}(\lambda, s, z) e(s) d\lambda ds dz = \sigma(\lambda, \Upsilon, Z) d\lambda d\Upsilon dZ$$

Here

$$e(s) = \frac{r_0^4}{(1-2sr_0^2)^2} \quad \text{for} \quad 0 \le 2r_0^2 s < 1$$

and

$$D_h = \{0 \le s \le h_\lambda(z), z \in [0,H], \lambda \in [0,2\pi]\} \qquad \text{with} \qquad 2h_\lambda(z) = \frac{1}{r_0^2} - \frac{1}{\varsigma_\lambda^2(z)}.$$

Let P be as in section 1.2 and assume we can choose Ψ such that $P_\lambda := P_\lambda(s,z)$ and $\Psi_\lambda := \Psi_\lambda(\Upsilon, Z)$ are legendre transforms of each other so that $s = \partial_Z \Psi_\lambda(\Upsilon, Z)$ and $z = \partial_\Upsilon \Psi_\lambda(\Upsilon, Z)$ for λ fixed. Assume in addition that (P, Ψ, h) satisfy

$$\begin{cases} \dfrac{r_0^4}{(1-2r_0^2\partial_\Upsilon \Psi_\lambda)^2} \det(\nabla^2_{\Upsilon,Z}\Psi) = \sigma_\lambda \\ \nabla_{\Upsilon,Z}\Psi_\lambda(spt(\sigma_\lambda)) = D_{h_\lambda} \\ P_\lambda(h_\lambda(z), z) = \frac{\Omega^2 r_0^2}{2(1-2r_0^2 h(\lambda,z))} \quad \text{on} \quad \{h_\lambda > 0\} \end{cases} \tag{1.4.1}$$

Then the Hamiltonian takes the form

$$\int \left(\frac{u^2}{2} + \frac{g}{\theta_0}\theta'\right)\chi_{\Gamma_\varsigma} r d\lambda dr dz = \int \left(-\Upsilon\partial_\Upsilon\Psi - Z\partial_Z\Psi + \frac{\Upsilon}{2r_0^2} + \Omega\sqrt{\Upsilon} + \frac{\Omega r^2}{2(1-2r_0^2\partial_\Upsilon\Psi)}\right)\sigma d\lambda d\Upsilon dZ \tag{1.4.2}$$

which makes it depend on σ and Ψ. We make the crucial observation that this Hamiltonian can be expressed solely in terms of the measure σ:

$$\mathbf{H}(\sigma) = \int_0^{2\pi} H_*(\sigma_\lambda) d\lambda \tag{1.4.3}$$

Here

$$H_*(\sigma_\lambda) := \inf_{\gamma \in \Gamma(e\chi_{D_{h_\lambda}}\mathcal{L}^2, \sigma_\lambda)} I(h_\lambda, \gamma) = \inf_{h_\lambda} \bar{I}[\sigma_\lambda](h_\lambda) \tag{1.4.4}$$

with

$$I(h,\gamma) := \int_{D_{h_\lambda}\times\mathbb{R}^2} \left(-s\Upsilon - zZ + \frac{\Upsilon}{2r_0^2} + \Omega\sqrt{\Upsilon} + \frac{\Omega r^2}{2(1-2r_0^2 s)}\right) d\gamma$$

and

$$\bar{I}[\sigma_\lambda](h_\lambda) = \frac{1}{2}W_2^2(\sigma, e(s)\chi_{D_{h_\lambda}}) + \int_{D_h} \left(\frac{\Omega^2 r_0^2}{2(1-2sr_0^2)} - \frac{s^2+z^2}{2} \right) dsdz$$
$$+ \int_{\mathbb{R}^2} \left(\frac{\Upsilon}{2r_0^2} - \Omega\sqrt{\Upsilon} - \frac{\Upsilon^2+Z^2}{2} \right) \sigma_\lambda(d\Upsilon, dZ)$$

The minimization problem in (1.4.4) has a dual formulation

$$H_*(\sigma_\lambda) = \sup J[\sigma_\lambda](\Psi, P) \tag{1.4.5}$$

$$J[\sigma_\lambda](\Psi, P) = \int_{\mathbb{R}^2} \Big(\frac{\Upsilon}{2r_0^2} - \Omega\sqrt{\Upsilon} - \Psi \Big) \sigma_\lambda(dq) + j(P) \tag{1.4.6}$$

with

$$j(P) = \inf_{h \in \mathcal{H}_0} \int_0^H \Pi_P(h(z), z) dz.$$

Here, $\mathcal{H}_0$ consists of all borel functions $h : [0, H] \longmapsto [0, 1/(2r_0^2)$. The supremum in (1.4.5) is taken over the set

$$\mathcal{U} := \Big\{ (\Psi, P) \in C(\mathbb{R}_+^2) \times C(\bar{\Delta}_{r_0}) : P(p) + \Psi(q) \geq \langle p, q \rangle \text{ for all } (p,q) \in \Delta_{r_0} \times \mathbb{R}_+^2 \Big\} \tag{1.4.7}$$

and

$$\Pi_P(\rho, z) = \int_0^\rho \Big(\frac{\Omega^2 r_0^2}{2(1-2sr_0^2)} - P(s,z) \Big) e(s) ds \quad \text{for} \quad 0 \leq 2r_0^2 \rho < 1. \tag{1.4.8}$$

It turns out that if h^σ is a minimizer in (1.4.4) and (P^σ, Ψ^σ) is a maximizer of (1.4.5) then $(h^\sigma, P^\sigma, \Psi^\sigma)$ solves (1.4.1).

1.5 Challenges.

The first challenge we encounter is to show that (1.4.1) admits a unique solution $(P^\sigma, \Psi^\sigma, h^\sigma)$. The lack of uniqueness would have been a source of a pessmistic pre-vision as for any hope of regularity. The difficulty here is that the set of functions

$\left\{\chi_{D_{h_\lambda}}(s,z)\right\}$ is not closed in the L^∞ weak* topology. This is an obstacle we bypass easily by observing that

$$\bar{I}[\sigma_\lambda](h_\lambda^{\#}) \leq \bar{I}[\sigma_\lambda](h_\lambda)$$

If $h_\lambda^{\#}$ is a monotone rearrangement of h_λ. The existence follows easily from the fact that the monotone functions are precompact with respect to pointwise topology. But the uniqueness proved extremely challenging in the sense that we don't know any strict convexity propriety for the functional with respect to any metric we could think of. We resort to a duality argument and discover a twist condition for a certain functional which ensure uniqueness.

The second challenge is to make rigorous the existence of $\mathbf{X}_\sigma$ in terms of $\nabla\Psi$. In other words we have to prove that $\left(\frac{\partial\Psi}{\partial\lambda}, \frac{\partial\Psi}{\partial\Upsilon}, \frac{\partial\Psi}{\partial Z}\right)$ exists. The existence of $\frac{\partial\Psi}{\partial\lambda}$ is equivalent to the regularity of the solution of the Monge Ampere equation

$$\frac{r_0^4}{(1-2r_0^2\partial_\Upsilon\Psi)^2}\det(\nabla^2_{\Upsilon,Z}\Psi) = \sigma_\lambda$$

with respect to a parameter λ when σ_λ depends smoothly on λ. This turns out to be a problem out of reach in this work, which we hope to address in a future study.

Strategy.

We decide to start the study of a system of equations which may not be physical but is educational and helps understand the original problem.

First simplication:

We remove all the expressions depending on λ from (1.0.2) (in particular, here $\frac{D}{Dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}$) to obtain the 2-dimensional system of equations.

$$
\begin{cases}
\frac{D\bar{u}}{Dt} + \frac{\bar{u}\bar{v}}{r} + 2\Omega\bar{v} = \frac{1}{r}\bar{F}, \quad \frac{\bar{u}^2}{r} + 2\Omega\bar{u} = \partial_r\bar{\varphi}, \quad \frac{D\bar{\theta}'}{Dt} = \bar{S}, \quad \text{in } \Gamma_{\bar{\varsigma}} \\
\\
\quad \frac{1}{r}\partial_r(r\bar{v}) + \partial_z\bar{w} = 0 \quad \partial_z\bar{\varphi} - g\frac{\bar{\theta}'}{\bar{\theta}_0} = 0, \quad \text{in } \Gamma_{\bar{\varsigma}} \\
\\
\partial_t\bar{\varsigma} + \bar{w}\partial_z\varsigma = \bar{v}, \qquad\qquad \text{on } \{r = \bar{\varsigma}\}
\end{cases} \tag{1.5.1}
$$

Here,

$$
\Gamma_{\bar{\varsigma}} := \{(r,z) \quad | \quad \bar{\varsigma}(z) \geq r \geq r_0, \;\; z \in [0,H]\},
$$

subject to the boundary condition

$$
\varphi(t, \bar{\varsigma}(t,H), H) = 0. \tag{1.5.2}
$$

Neumann condition has been imposed on the rigid boundary.

In the “dual space”, this system of PDEs takes the form

$$
\begin{cases}
\frac{\partial\sigma}{\partial t} + \operatorname{div}(V_t[\sigma]\sigma) = 0 \qquad (0,T) \times \mathbb{R}^2 \\
\\
\sigma|_{t=0} = \sigma_0.
\end{cases}
$$

and has been studied under two different conditions namely when the initial data is absolutely continuous with respect to Lebesgue and when it is not.
We note that the collection of variational problems stemming from the 2-dimensional system coincides with the one obtained from our original problem.

1.6 Plan of this work.

In Chapter II, we first collect the notations used throughout this manuscript and recall a few definitions. We show how the various systems of equations we study are derived. Next, we lay down a program whose completion will solve the Almost Axisymmetric Flows with Forcing Terms and explain their relationship to a collection of variational problems.

Chapter III contains our main contribution from the calculus of variations point of view. The uniqueness of a minimizer satisfying the boundary condition (1.4.1) (iii) along with the Lipschitz regularity of the boundary of D_{h_λ} are certainly the most remarkable facts.

In Chapter IV, It is worth mentioning that we have been able to give a meaning to the velocity field $V_t[\sigma]$ even for measures which are not absolutely continuous, based on the Riesz representation argument. Having all these tools at hand, existence of a solution in (4.2.6) based on a scheme where $V_t[\sigma]$ is implicitly defined, is by now standard and pionnered by Ambrosio and Gangbo [2].

We end this manuscript with an Appendix which consists of basic Analysis results.

CHAPTER II

DERIVATIONS OF DIVERSE SYSTEMS

2.1 Preliminaries.

In this section we introduce some notations and definitions.

- For any real number x, $[x]$ denotes the integer part of x.

-Let $\mathbf{x} = (x_1, x_2...x_n)$ and $\mathbf{y} = (y_1, y_2...y_n)$ be vectors in $\mathbb{R}^n$, n an integer greater or equal to 1. Then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}.\mathbf{y} = \sum_{k=1}^{k=n} x_k y_k.$$

-If $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, ...\mathbf{A}_n)$ with $\mathbf{A}_i \in \mathbb{R}^k$ and $\mathbf{x} \in \mathbb{R}^k$ then

$$< \mathbf{x}, \mathbf{A} >= (< \mathbf{x}, \mathbf{A}_1 >, ..., < \mathbf{x}, \mathbf{A}_{n-1} >, < \mathbf{x}, \mathbf{A}_n >)$$

- Let $\mathbf{v} = (v_1, v_2, v_3)$ in the basis $(\partial_a,\ \partial_b,\ \partial_c)$ associated with system of coordinates (a, b, c). Then $\mathrm{div}(\mathbf{v}) = \partial_a v_1 + \partial_b v_2 + \partial_b v_3$. This is an abuse of notation but note that if (a, b, c) is the cartesian coordinates then $\mathrm{div}(\mathbf{v})$ is the expression of divergence of $\mathbf{v}$.

- Let $\mathbf{v} \in \mathbb{R}^n$. We denote by $\mathbf{t_v}$ the translation of $\mathbf{v}$ defined by $\mathbf{t_v}(\mathbf{u}) = \mathbf{u} + \mathbf{v}$ for any $\mathbf{u} \in \mathbb{R}^n$.

-Let M be a matrix. M^T denotes the transpose of M and if M is invertible then we denote the inverse of the transpose of M by M^{-T}.

- Let A and B be two subsets of $\mathbb{R}^n$ $n \geq 1$. Then the difference symmetric of A and B is denoted by $A \Delta B$ and is equal to $(A \setminus B) \cup (B \setminus A)$.

- For convenience, we use the notation $S_t := S(t,\cdot,\cdot,\cdot)$ and $S_t^\lambda := S(t,\lambda,\cdot,\cdot)$.

-$\frac{D}{Dt}$ denotes the material derivative and will be defined in each context as we make precise the velocity field it corresponds to.

If $F = (F_1, F_2, ...F_n)$ is smooth then $\frac{DF}{Dt} = \left(\frac{DF_1}{Dt}, \frac{DF_2}{Dt}, ... \frac{DF_n}{Dt}\right)$.

- If $f : A \longmapsto \mathbb{R}$ ($A \subset \mathbb{R}^n$) is a Lipschitz continuous then $Lip(f) \equiv \inf_{\substack{\mathbf{x},\mathbf{y}\in A \\ \mathbf{x}\neq\mathbf{y}}} \dfrac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|}$ denotes the lipschitz constant of f.

-H^d denotes the d-dimensional Hausdorff measure.

- $\mathcal{P}_p(\mathbb{R}^d)$ denotes the set of all probabilities with a finite p-moment.

- Given two borel μ_0 and μ_1 be borel finite measures in $\mathbb{R}^d$, we denote by $\Gamma(\mu_0,\mu_1)$ the set of all measures on $\mathbb{R}^d \times \mathbb{R}^d$ whose first and second marginal are respectively μ_0 and μ_1.

Definition 2.1.1 *Let μ_0 and μ_1 be borel measures in $\mathbb{R}^d$. The (p-th) Wasserstein distance between the measures μ_0 and μ_1 elements of $\mathcal{P}_p(\mathbb{R}^d)$ is defined by*

$$W_p^p(\mu_0,\mu_1) = \begin{cases} \min\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p d\gamma : \gamma \in \Gamma(\mu_0,\mu_1)\right\} & \textit{if } \mu_0(\mathbb{R}^d) = \mu_1(\mathbb{R}^d) < \infty \\ \infty \qquad \textit{otherwise} . & \end{cases} \tag{2.1.1}$$

The set of minimizers in (2.1.1) is denoted by $\Gamma_0(\mu_0,\mu_1)$.

Definition 2.1.2 *Let μ_0 and μ_1 be borel finite measures on $\mathbb{R}^d$ such that $\mu_0(\mathbb{R}^d) = \mu_1(\mathbb{R}^d)$. We say that a borel map T pushes forward μ_0 onto μ_1 and write $T_\#\mu_0 = \mu_1$ if*

$$\int_{\mathbb{R}^d} F d\mu_1 = \int_{\mathbb{R}^d} F \circ T d\mu_0 \tag{2.1.2}$$

for all continuous and bounded functions F on $\mathbb{R}^d$. The set of minimizers in (2.1.1) is denoted by $\Gamma_0(\mu_0,\mu_1)$

2.2 Derivation of the almost axisymmetric flow equations with forcing terms.

In this paragraph, we show how the equations in (1.0.2) are derived from the Boussinesq equations with forcing terms, expressed in cartesian coordinates (x, y, z). As mentioned in the introduction, the Boussinesq equations with forcing term $\bar{\mathbf{F}}$ are given by

$$\begin{cases} \frac{\bar{D}\bar{\mathbf{u}}}{Dt} + 2\Omega\bar{\mathbf{u}} \times \partial_z + \nabla\bar{\varphi} - \frac{g}{\theta_0}\theta\partial_z - \bar{\mathbf{F}} = \mathbf{0} \\ \frac{\bar{D}\bar{\theta}}{Dt} = \bar{S} \\ \operatorname{div}(\bar{\mathbf{u}}) = 0 \end{cases} \tag{2.2.1}$$

Here, the velocity field $\bar{\mathbf{u}} = (\bar{u}, \bar{v}, \bar{w})^T$ is expressed in the basis $(\partial_x, \partial_y, \partial_z)$ corresponding to (x, y, z).

We consider the change of coordinates $\mathcal{P}$ from the cylindrical coordinates (λ, r, z) to (x, y, z) defined by

$$\mathcal{P}(\lambda, r, z) := (r\cos(\lambda), r\sin(\lambda), z)$$

To $\bar{\mathbf{u}} = (\bar{u}, \bar{v}, \bar{w})^{\mathrm{T}}$, we associate the corresponding velocities $\mathbf{u} = (u, v, w)^{\mathrm{T}}$ in the orthonormal basis $(\frac{1}{r}\partial_\lambda, \partial_r, \partial_z)$ (with respect to the metric $rd\lambda^2 + dr^2 + dz^2$) and $\mathbf{u_c} = (\frac{u}{r}, v, w)^{\mathrm{T}}$ in the basis $(\partial_\lambda, \partial_r, \partial_z)$ both associated to the coordinate system (λ, r, z). $\bar{\mathbf{u}}$ and $\mathbf{u_c}$ are related by

$$[\nabla\mathcal{P}]\mathbf{u}_c = \bar{\mathbf{u}} \circ \mathcal{P} \tag{2.2.2}$$

We recall

$$\frac{\bar{D}}{Dt} = \frac{\partial}{\partial t} + \langle \bar{\mathbf{u}},\ \nabla_{x,y,z}\cdot\rangle \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \langle \mathbf{u}_c,\ \nabla_{\lambda,r,z}\cdot\rangle$$

We set

$$\mathbb{P}(\lambda) = \begin{pmatrix} -\sin\lambda & \cos\lambda & 0 \\ \cos\lambda & \sin\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbb{S}(r) = \begin{pmatrix} r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and observe that

$$\mathbb{S}(r)\mathbf{u_c}(\lambda, r, z) = \mathbf{u}(\lambda, r, z) \qquad \mathbb{P}^{-1}(\lambda) = \mathbb{P}(\lambda) \qquad \mathbb{P}^{T}(\lambda) = \mathbb{P}(\lambda) \qquad \mathbb{S}^{-1}(r) = \mathbb{S}(\frac{1}{r})$$

Note that

$$\frac{D}{Dt}\mathbb{S}(\frac{1}{r}) = \begin{pmatrix} -\frac{v}{r^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since

$$\nabla \mathcal{P}(\lambda, r, z) = \begin{pmatrix} -r\sin\lambda & \cos\lambda & 0 \\ r\cos\lambda & \sin\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We easily check that

$$\nabla \mathcal{P}(\lambda, r, z) = \mathbb{P}(\lambda)\mathbb{S}(r) \tag{2.2.3}$$

We note that for any real-valued smooth function $\bar{m} := \bar{m}(t, x, y, z)$

$$\begin{aligned} \frac{D}{Dt}(\bar{m} \circ \mathcal{P}) &= \frac{\partial \bar{u}}{\partial t} \circ \mathcal{P} + \langle \mathbf{u_c},\ \nabla(\bar{m} \circ \mathcal{P})\rangle \\ &= \frac{\partial \bar{m}}{\partial t} \circ \mathcal{P} + \langle \mathbf{u_c},\ [\nabla \mathcal{P}]^T \nabla \bar{m} \circ \mathcal{P}\rangle \\ &= \frac{\partial \bar{m}}{\partial t} \circ \mathcal{P} + \langle [\nabla \mathcal{P}]\mathbf{u_c},\ \nabla \bar{m} \circ \mathcal{P}\rangle \\ &= \frac{\partial \bar{m}}{\partial t} \circ \mathcal{P} + \langle \bar{\mathbf{u}} \circ \mathcal{P},\ \nabla \bar{m} \circ \mathcal{P}\rangle \\ &= \frac{\bar{D}\bar{m}}{Dt} \circ \mathcal{P} \end{aligned} \tag{2.2.4}$$

Applying (2.2.4) to $\bar{u},\ \bar{v},\ \bar{w}$, we obtain

$$\frac{D}{Dt}(\bar{\mathbf{u}} \circ \mathcal{P}) = \frac{\bar{D}\bar{\mathbf{u}}}{Dt} \circ \mathcal{P} \tag{2.2.5}$$

and so, by using (2.2.2) and (2.2.4) we obtain

$$\frac{\bar{D}\bar{\mathbf{u}}}{Dt} \circ \mathcal{P} = \frac{D}{Dt}(\bar{\mathbf{u}} \circ \mathcal{P}) = \frac{D}{Dt}([\nabla \mathcal{P}]\mathbf{u_c}) = \frac{D}{Dt}([\nabla \mathcal{P}])\mathbf{u_c} + [\nabla \mathcal{P}]\frac{D\mathbf{u_c}}{Dt} \tag{2.2.6}$$

We compute

$$\frac{D}{Dt}([\nabla\mathcal{P}]) = \begin{bmatrix} -(u\cos\lambda + v\sin\lambda) & -\frac{u}{r}\sin\lambda & 0 \\ (-u\sin\lambda + v\cos\lambda) & \frac{u}{r}\cos\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and notice that

$$\frac{D}{Dt}([\nabla\mathcal{P}]) = \mathbb{P}(\lambda)\begin{bmatrix} v & \frac{u}{r} & 0 \\ -u & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so

$$\frac{D}{Dt}([\nabla\mathcal{P}])\mathbf{u_c} = \mathbb{P}(\lambda)\left(\frac{2uv}{r}, -\frac{u^2}{r}, 0\right)^T \tag{2.2.7}$$

As $\mathbf{u_c}(\lambda, r, z) = \mathbb{S}(\frac{1}{r})\mathbf{u}(\lambda, r, z)$, we have

$$\begin{aligned} \frac{D\mathbf{u_c}}{Dt} &= \mathbb{S}(\frac{1}{r})\frac{D\mathbf{u}}{Dt} + \frac{D\mathbb{S}(\frac{1}{r})}{Dt}\mathbf{u} \\ &= \mathbb{S}(\frac{1}{r})\frac{D\mathbf{u}}{Dt} - (\frac{uv}{r^2}, 0, 0)^T \end{aligned}$$

This, combined with (2.2.3) yields

$$[\nabla\mathcal{P}]\frac{D\mathbf{u_c}}{Dt} = \mathbb{P}(\lambda)\left[\frac{D\mathbf{u}}{Dt} - \mathbb{S}(r)(\frac{uv}{r^2}, 0, 0)^T\right] = \mathbb{P}(\lambda)\left[\frac{D\mathbf{u}}{Dt} - (\frac{uv}{r}, 0, 0)^T\right] \tag{2.2.8}$$

We combine (2.2.6), (2.2.7) and (2.2.8) to get

$$\frac{\bar{D}\bar{\mathbf{u}}}{Dt}\circ\mathcal{P} = \mathbb{P}(\lambda)\left[\frac{D\mathbf{u}}{Dt} + \left(\frac{uv}{r}, -\frac{u^2}{r}, 0\right)\right] \tag{2.2.9}$$

We use (2.2.2), the fact that $\mathbb{S}(r)\mathbf{u_c}(\lambda, r, z) = \mathbf{u}(\lambda, r, z)$ and (2.2.3) to obtain that

$$(\bar{\mathbf{u}}\times\partial_z)\circ\mathcal{P} = (\bar{\mathbf{u}}\circ\mathcal{P})\times\partial_z = ([\nabla\mathcal{P}]\mathbf{u_c})\times\partial_z = (\mathbb{P}(\lambda)\mathbb{S}(r)\mathbf{u_c})\times\partial_z = (\mathbb{P}(\lambda)\mathbf{u})\times\partial_z \tag{2.2.10}$$

We check that

$$(\mathbb{P}(\lambda)\mathbf{u})\times\partial_z = \mathbb{P}(\lambda)(\mathbf{u}\times\partial_z) \tag{2.2.11}$$

We combine (2.2.10) and (2.2.11) to get

$$(\bar{\mathbf{u}} \times \partial_z) \circ \mathcal{P} = \mathbb{P}(\lambda)(\mathbf{u} \times \partial_z) = \mathbb{P}(\lambda)(v, -u, 0)^T \tag{2.2.12}$$

Setting

$$\varphi = \bar{\varphi} \circ \mathcal{P}$$

We get

$$(\nabla\bar{\varphi}) \circ \mathcal{P} = (\nabla\mathcal{P})^{-\mathrm{T}}\nabla\varphi = (\mathbb{P}(\lambda)\mathbb{S}(r))^{-\mathrm{T}}\nabla\varphi = \mathbb{P}(\lambda)[\mathbb{S}(\frac{1}{r})\nabla\varphi] \tag{2.2.13}$$

Set

$$\theta = \bar{\theta} \circ \mathcal{P}$$

Note that

$$\partial_z = \mathbb{P}\partial_z$$

Combine (2.2.9), (2.2.12) (2.2.13) to get

$$\begin{aligned}&\left[\frac{D\bar{\mathbf{u}}}{Dt} + 2\Omega\bar{\mathbf{u}} \times \partial_z + \nabla\bar{\varphi} - \frac{g}{\theta_0}\bar{\theta}\partial_z - \bar{\mathbf{F}}\right] \circ \mathcal{P} \\ &= \mathbb{P}(\lambda)\left[\frac{D\mathbf{u}}{Dt} + (\frac{uv}{r} + 2\Omega v, -\frac{u^2}{r} - 2\Omega u, -\frac{g}{\theta_0}\theta)^T + \mathbb{S}(\frac{1}{r})\nabla\varphi - \mathbb{P}^{-1}\bar{\mathbf{F}} \circ \mathcal{P}\right]\end{aligned} \tag{2.2.14}$$

As $\det(\mathbb{P}) = -1$, in view of (2.2.14), the first equation in (2.2.1) is equivalent to

$$\frac{D\mathbf{u}}{Dt} + (\frac{uv}{r} + 2\Omega v, -\frac{u^2}{r} - 2\Omega u, -\frac{g}{\theta_0}\theta)^T + \mathbb{S}(\frac{1}{r})\nabla\varphi - \mathbb{P}^{-1}\bar{\mathbf{F}} \circ \mathcal{P} = \mathbf{0} \tag{2.2.15}$$

This equation is written explicitly in (1.1.2) when

$$\mathbf{F} := \mathbb{P}^{-1}\bar{\mathbf{F}} \circ \mathcal{P} = (\frac{1}{r}F, 0, 0)^T.$$

By setting $S = \bar{S} \circ \mathcal{P}$, we use (2.2.4) to obtain

$$\frac{D\theta}{Dt} = \frac{D}{Dt}(\bar{\theta} \circ \mathcal{P}) = \frac{\bar{D}\bar{\theta}}{Dt} \circ \mathcal{P} = \bar{S} \circ \mathcal{P} = S \tag{2.2.16}$$

It is well known in the litterature that (divergence expressed in cylindrical coordinates)

$$\operatorname{div}(\bar{\mathbf{u}}) = \frac{1}{r}\operatorname{div}(r\mathbf{u}_c)$$

Therefore the second equation in (2.2.1) becomes

$$0 = \frac{1}{r}\frac{\partial}{\partial\lambda}(u) + \frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{\partial w}{\partial z} \tag{2.2.17}$$

Thus, (2.2.15), (2.2.16) and (2.2.17) form the system of equations (1.1.2).

2.3 Derivation of the problem in the dual space

Let $\mathbf{u} = (u, v, w)$ be a velocity field in the cylindrical coordinates in the basis $(\frac{1}{r}\partial_\lambda, \partial_r, \partial_z)$. Recall that we have defined by

$$\mathbf{u_c} = (\frac{u}{r}, v, w)$$

the corresponding velocity field in the basis $(\partial_\lambda, \partial_r, \partial_z)$. We assume that the velocity field is smooth in $\{r > r_0\}$.

2.3.1 Velocity in the dual space.

Let N be the flow in cartesian coordinates corresponding to $\mathbf{u_c}$ defined by

$$\dot{N}_t = \mathbf{u_{ct}} \circ N_t. \qquad N_0 = \mathbf{id} \qquad t \in (0, T_*) \tag{2.3.1}$$

and ϕ be another flow associated to a velocity field $\mathbf{X}$ defined by

$$\dot{\phi}_t = \mathbf{X}_t \circ \phi_t. \qquad \phi_0 = \mathbf{id} \qquad t \in (0, T_*) \tag{2.3.2}$$

We consider a function $\mathfrak{F} : [0, T_*] \times [0, 2\pi] \times \mathbb{R}^2 \longmapsto \mathbb{R}^3$. We assume that $\mathfrak{F}$ is smooth on $(0, T_*) \times (0, 2\pi) \times \mathbb{R}^2$ such that $\mathfrak{F}_t := \mathfrak{F}(t, \cdot)$ is invertible for all $t \in [0, T_*]$ such that N and ϕ are related by

$$\phi_t = \mathfrak{F}_t \circ N_t \circ \mathfrak{F}_0^{-1} \tag{2.3.3}$$

To obtain an explicit expression of $\mathbf{X}$ in terms of $\mathfrak{F}$ and $\mathbf{u_c}$, we combine (2.3.1), (2.3.2) and (2.3.3) to get

$$\begin{aligned}
\dot{\phi}_t &= \partial_t(\mathfrak{F}_t \circ N_t \circ \mathfrak{F}_0^{-1}) \\
&= (\partial_t \mathfrak{F}_t) \circ N_t \circ \mathfrak{F}_0^{-1} + \langle \nabla \mathfrak{F}_t \circ N_t \circ \mathfrak{F}_0^{-1}, \dot{N}_t \circ \mathfrak{F}_0^{-1} \rangle \\
&= (\partial_t \mathfrak{F}_t + \langle \nabla \mathfrak{F}_t, \mathbf{u_c}_t \rangle) \circ N_t \circ \mathfrak{F}_0^{-1} \\
&= (\partial_t \mathfrak{F}_t + \langle \nabla \mathfrak{F}_t, \mathbf{u_c}_t \rangle) \circ \mathfrak{F}_t^{-1} \circ \phi_t
\end{aligned}$$

so that

$$\mathbf{X} = \dot{\phi}_t \circ \phi_t^{-1} = (\partial_t \mathfrak{F}_t + \langle \nabla \mathfrak{F}_t, \mathbf{u_c}_t \rangle) \circ \mathfrak{F}_t^{-1}$$

When we interchange the role of $\mathbf{u}$ and $\mathbf{X}$ in the above reasoning,and use $\mathfrak{F}_t^{-1}$ instead of $\mathfrak{F}_t$ we obtain equivalently

$$\mathbf{u_c}_t = \left(\partial_t \mathfrak{F}_t^{-1} + \langle \nabla \mathfrak{F}_t^{-1}, \mathbf{X}_t \rangle\right) \circ \mathfrak{F}_t$$

2.3.2 Domain and normal vectors.

Let $\varsigma : [0,T] \times [0,2\pi] \times [0,H] \longmapsto \mathbb{R}$ be a function. Define

$$\mathcal{K} = \{(t,\lambda,r,z) : 0 < t < T,\ (\lambda,r,z) \in \Gamma_{\varsigma_t}\} \tag{2.3.4}$$

where

$$\Gamma_{\varsigma_t} = \{(\lambda,r,z) : 0 \leq \lambda \leq 2\pi,\ \ 0 \leq z \leq H,\ \ r_0 \leq r \leq \varsigma_t(\lambda,z)\}$$

We write

$$\partial\mathcal{K} = \mathcal{K}^{sp} \cup \mathcal{K}^{ti}$$

Here

$\mathcal{K}^{sp} = \{0\} \times \Gamma_{\varsigma_0} \cup \{T\} \times \Gamma_{\varsigma_T}$ and $\mathcal{K}^{ti} = \{(t,\lambda,r,z) : 0 < t < T;\ (\lambda,r,z) \in \partial\Gamma_{\varsigma_t}\}$

we note that the boundary $\partial\Gamma_{\varsigma_t}$ of Γ_{ς_t} is the union of the subsets $\{z=0\}$, $\{z=H\}$, $\{r=r_0\}$ and $\{r=\varsigma_t(\lambda,z)\}$ of Γ_{ς_t} in $\mathbb{R}^3$ for each t fixed. And so

$$\mathcal{K}^{ti} = \{z=0\}_* \cup \{z=H\}_* \cup \{r=r_0\}_* \cup \{r=\varsigma(t,\lambda,z)\}_* \tag{2.3.5}$$

in time-space. The star (*) in subscript in the above equation specifies that the sets are considered in $\mathbb{R}^4$. We parametrize the boundary $\mathcal{K}^{ti}$ of the domain $\mathcal{K}$ and obtain that if ς is smooth, the outward unit normal vector in the time-space is given by

$$\underline{\mathbf{n}} = \begin{cases} (\partial_t\varsigma_t, \partial_\lambda\varsigma_t, -1, \partial_z\varsigma_t))/\sqrt{|\nabla_{t,\lambda,z}\varsigma_t|)^2+1} & \text{on } \{r = \varsigma(t,\lambda,z)\}_*\,, \\ (0,0,1,0) & \text{on } \{r = r_0\}_*\,, \\ (0,0,0,1) & \text{on } \{z = H\}_* \\ (0,0,0,-1) & \text{on } \{z = 0\}_* \end{cases} \tag{2.3.6}$$

in time-space.

Remark 2.3.1 *Assume ς is continuous and $r_0 < \varsigma$. Then ς has a minimum value a_0 on his domain such that $r_0 < a_0$. Consider a bump function ϕ_0 such that*

$$0 < \phi_0 < a_0 - r_0 \qquad \textit{on} \qquad (0,H)$$

and assume the value 0 elsewhere. Set

$$\phi(t,\lambda,z) = \phi_0(z)$$

on $[0,T]\times[0,2\pi]\times[0,H]$. We define

$$\rho_1 = \varsigma - \phi \qquad \textit{and} \qquad \rho_2 = \varsigma + \phi$$

Then

$$\begin{cases} r_0 < \rho_1 < \varsigma < \rho_2 & \{(t,\lambda,z) : t \in [0,T],\ \lambda \in [0,2\pi]\ z \in (0,H)\} \\ \rho_1(t,\lambda,0) = \rho_2(t,\lambda,0) = \varsigma(t,\lambda,0) & \{(t,\lambda,0) : t \in [0,T],\ \lambda \in [0,2\pi]\} \\ \rho_1(t,\lambda,H) = \rho_2(t,\lambda,H) = \varsigma(t,\lambda,H) & \{(t,\lambda,H) : t \in [0,T],\ \lambda \in [0,2\pi]\} \end{cases} \tag{2.3.7}$$

Set

$$\mathcal{J} := \{(t,\lambda,r,z) : 0 < t < T,\ 0 < \lambda < 2\pi,\ 0 < z < H,\ \rho_1(t,\lambda,z) < r < \rho_1(t,\lambda,z)\}$$

We note that

$$\mathcal{J} \cap \{z=0\}_* = \mathcal{J} \cap \{z=H\}_* = \mathcal{J} \cap \{r=r_0\}_* = \emptyset \tag{2.3.8}$$

and

$$\{r=\varsigma\}_* \subset \bar{\mathcal{J}}$$

Observe that

$$\{r=\varsigma\}_* = \{r=\varsigma\}_* \cap \bar{\mathcal{J}} = (\{r=\varsigma\}_* \cap \mathcal{J}) \cup (\{r=\varsigma\}_* \cap \partial\mathcal{J}) \tag{2.3.9}$$

Here, $\bar{\mathcal{J}}$ *and* $\partial\mathcal{J}$ *denote respectively the closure and boundary of* $\mathcal{J}$. *But*

$$\begin{aligned} \{r=\varsigma\}_* \cap \partial\mathcal{J} = &\{(t,\lambda,\varsigma(t,\lambda,0),0) : 0 \le t \le T,\ 0 \le \lambda \le 2\pi\} \\ &\cup \{(t,\lambda,\varsigma(t,\lambda,0),0) : 0 \le t \le T,\ 0 \le \lambda \le 2\pi\} \end{aligned} \tag{2.3.10}$$

We note that the sets on the right handside of (2.3.10) are graphs of 2- dimensional surfaces and so

$$\mathrm{H}^3(\{r=\varsigma\}_* \cap \partial\mathcal{J}) = 0 \tag{2.3.11}$$

2.3.3 Conservation of the total mass.

We recall that $\mathbf{u} = (u, v, w)$ is a smooth velocity field in the cylindrical coordinates in the basis $(\frac{1}{r}\partial_\lambda, \partial_r, \partial_z)$ and $\mathbf{u_c} = (\frac{u}{r}, v, w)$ is the corresponding velocity field in the basis $(\partial_\lambda, \partial_r, \partial_z)$. We consider the following system of equations

$$\begin{cases} \operatorname{div}(r\mathbf{u_c}) = 0 \ \ \textit{on} \ \ \Gamma_{\varsigma_t} \\ \mathbf{u}_t \cdot \mathbf{n}_t = 0 \ \ \{z=0\} \cup \{z=H\} \cup \{r=r_0\} \\ \frac{\partial \varsigma_t}{\partial t} + \frac{u}{r}\frac{\partial \varsigma_t}{\partial \lambda} + w\frac{\partial \varsigma_t}{\partial z} = v \qquad \textit{on} \ \ \{r=\varsigma(t,\lambda,z)\}. \end{cases} \tag{2.3.12}$$

Here, $\mathbf{n}_t$ is outward unit normal vector of $\partial\Gamma_{\varsigma_t}$ for each t fixed and the equations in (2.3.12) express the conservation of the mass in the physical space for the almost axisymmetric flows.

Let $\{\sigma_t\}_{t\in(0,T)}$ be a borel family of probabilities and $\mathbf{X}_t$ be a velocity field solving the continuity equation

$$\frac{\partial\sigma_t}{\partial t} + \mathrm{div}(\mathbf{X}\sigma_t) = 0 \qquad (0,T)\times\mathbb{R}^3. \tag{2.3.13}$$

in the sense of distributions.

Remark 2.3.2 *(i) we note that*

$$\mathrm{div}(r\mathbf{u}_c) = r\left[\frac{1}{r}\frac{\partial}{\partial\lambda}(u) + \frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{\partial w}{\partial z}\right]$$

(ii) for each t fixed, the normal vector in space given by

$$\mathbf{n_t} = \begin{cases} (0,1,0) & \text{on } \{r = r_0\}, \\ (0,0,1) & \text{on } \{z = H\} \\ (0,0,-1) & \text{on } \{z = 0\} \end{cases} \tag{2.3.14}$$

in either $(\frac{1}{r}\partial_\lambda, \partial_r, \partial_z)$ or $(\partial_\lambda, \partial_r, \partial_z)$. So, using the corresponding metrics, we easily check that

$$\mathbf{u_c}_t \cdot \mathbf{n_t} = \mathbf{u_t}\cdot\mathbf{n_t}$$

Lemma 2.3.3 *Assume for simplicity that ς is smooth and $r_0 < \varsigma$. Let $\mathfrak{F}$ be a smooth function on $(0,T)\times(0,2\pi)\times\mathbb{R}^2$ such that $\mathfrak{F}_t$ invertible for all $t\in[0,T]$ and satisfy*

$$\mathfrak{F}_{t\#}(r\chi_{\Gamma_{\varsigma_t}}) = \sigma_t \qquad \text{and} \qquad \mathbf{X}\circ\mathfrak{F}_t = \frac{\partial\mathfrak{F}_t}{\partial t} + \langle\nabla\mathfrak{F}_t, \mathbf{u_c}\rangle \tag{2.3.15}$$

Then (2.3.12) and (2.3.13) are equivalent.

Proof: Assume $\mathfrak{F}_t$ satisfies (2.3.15). Let $\mathcal{K}$ be as in (2.3.4), N and ϕ as in (2.3.1) and (2.3.2). Let ζ and $\psi \in C_c^1((0,T)\times\mathbb{R}^3)$ such that

$$\zeta_t = \psi_t \circ \mathfrak{F}_t$$

On the one hand, in view of (2.3.2) and (2.3.3)

$$\begin{aligned}\partial_t(\zeta_t \circ N_t) &= \partial_t(\psi_t \circ \mathfrak{F}_t \circ N_t)\\ &= \partial_t(\psi_t \circ \phi_t \circ \mathfrak{F}_0)\\ &= \frac{\partial \psi}{\partial t} \circ \phi_t \circ \mathfrak{F}_0 + \langle \nabla\psi \circ \phi_t \circ \mathfrak{F}_0, \dot{\phi}_t \circ \mathfrak{F}_0\rangle \\ &= \frac{\partial \psi}{\partial t} \circ \phi_t \circ \mathfrak{F}_0 + \langle \nabla\psi, \mathbf{X}\rangle \circ \phi_t \circ \mathfrak{F}_0\\ &= \frac{\partial \psi}{\partial t} \circ \mathfrak{F}_t \circ N_t + \langle \nabla\psi, \mathbf{X}\rangle \circ \mathfrak{F}_t \circ N_t\end{aligned} \tag{2.3.16}$$

On the other hand, (2.3.1) yields

$$\partial_t(\zeta_t \circ N_t) = \frac{\partial \zeta_t}{\partial t} \circ N_t + \langle \nabla\zeta_t, \mathbf{u_c}\rangle \circ N_t \tag{2.3.17}$$

We conclude from (2.3.16) and (2.3.17) that

$$\big(\frac{\partial \zeta_t}{\partial t} + \langle \nabla\zeta_t, \mathbf{u_c}\rangle\big) \circ \mathfrak{F}_t^{-1} = \frac{\partial \psi_t}{\partial t} + \langle \nabla\psi, \mathbf{X}\rangle$$

Thus,

$$\int_0^T \int_{\mathbb{R}^3} (\frac{\partial \psi_t}{\partial t} + \langle \mathbf{X}, \nabla\psi_t\rangle) d\sigma_t dt = \int_0^T \int_{\mathbb{R}^3} (\frac{\partial \zeta_t}{\partial t} + \langle \mathbf{u_c}, \nabla\zeta\rangle) \circ \mathfrak{F}_t^{-1} d\sigma_t dt$$

As $\mathfrak{F}_t$ is invertible, smooth and $\mathfrak{F}_{t\#} r\chi_{\Gamma_{\varsigma_t}} = \sigma_t$, we obtain that $\mathfrak{F}_{t\#}^{-1}\sigma_t = r\chi_{\Gamma_{\varsigma_t}}$ and so

$$\int_0^T \int_{\mathbb{R}^3} (\frac{\partial \psi_t}{\partial t} + \langle \mathbf{X}, \nabla\psi_t\rangle) d\sigma_t dt = \int_0^T \int_{\mathbb{R}^3} \frac{\partial \zeta_t}{\partial t} + \langle \mathbf{u_c}, \nabla\zeta\rangle d(r\chi_{\Gamma_{\varsigma_t}}) dt$$

That is,

$$\int_0^T \int_{\mathbb{R}^3} (\frac{\partial \psi_t}{\partial t} + \langle \mathbf{X}, \nabla\psi_t\rangle) d\sigma_t dt = \iiiint_{\mathcal{K}} r\frac{\partial \zeta_t}{\partial t} + \langle r\mathbf{u_c}, \nabla\zeta\rangle d\lambda dr dz dt \tag{2.3.18}$$

We apply the divergence theorem in the time-space to obtain

$$\iiiint_{\mathcal{K}} r\frac{\partial \zeta_t}{\partial t} + \langle r\mathbf{u_c}, \nabla\zeta\rangle d\lambda dr dz dt = -\iiiint_{\mathcal{K}} \zeta\underline{\mathrm{div}}(r, r\mathbf{u_c}) d\lambda dr dz dt + \iiint_{\partial\mathcal{K}} \zeta\langle r(1,\mathbf{u_c}), \underline{\mathbf{n}}\rangle d\mathrm{H}^3 \tag{2.3.19}$$

$\underline{\mathrm{div}}$ denotes the time-space divergence in cartesian coordinates. Note that $\zeta = 0$ on $\mathcal{K}^{sp}$. Using (2.3.6), we obtain

$$\langle(1,\mathbf{u_c}), \underline{\mathbf{n}}\rangle = \mathbf{u_c}_t \cdot \mathbf{n}_t = \mathbf{u}_t \cdot \mathbf{n}_t \quad \{z=0\}_* \cup \{z=H\}_* \cup \{r=r_0\}_* \tag{2.3.20}$$

and

$$\langle(1,\mathbf{u_c}), \underline{\mathbf{n}}\rangle = (\partial_t\varsigma_t + \frac{u}{r}\partial_\lambda\varsigma_t + \partial_z\varsigma_t - v)/\sqrt{|\nabla_{t,\lambda,z}\varsigma_t|^2 + 1} \qquad \text{on } \{r = \varsigma(t,\lambda,z)\}_* \tag{2.3.21}$$

We next compute the second term in the left hand side of (2.3.19).

$$\begin{aligned}
&\iiint_{\partial\mathcal{K}} \zeta\langle r(1,\mathbf{u_c}),\underline{\mathbf{n}}\rangle d\mathrm{H}^3 \\
&= \iiint_{\mathcal{K}^{sp}} \zeta\langle r(1,\mathbf{u_c}), \underline{\mathbf{n}}\rangle d\mathrm{H}^3 + \iiint_{\mathcal{K}^{ti}} \zeta\langle r(1,\mathbf{u_c}), \underline{\mathbf{n}}\rangle d\mathrm{H}^3 \\
&= \iiint_{\mathcal{K}^{ti}} \zeta\langle r(1,\mathbf{u_c}), \underline{\mathbf{n}}\rangle d\mathrm{H}^3 \\
&= \iiint_{\{z=0\}_*\cup\{z=H\}_*\cup\{r=r_0\}_*} r\zeta\mathbf{u}_t \cdot \mathbf{n}_t d\mathrm{H}^3 \\
&\quad + \iiint_{\{r=\varsigma(t,\lambda,z)\}_*} \frac{\varsigma\zeta}{\sqrt{|\nabla_{t,\lambda,z}\varsigma_t|^2+1}}(\partial_t\varsigma_t + \frac{u}{r}\partial_\lambda\varsigma_t + w\partial_z\varsigma_t - v) d\mathrm{H}^3
\end{aligned} \tag{2.3.22}$$

The second equality is due to the fact that $\zeta = 0$ on $\mathcal{K}^{sp}$. The third equality comes from (2.3.20) and (2.3.21). We observe that $\underline{\mathrm{div}}(r, r\mathbf{u_c}) = \mathrm{div}(r\mathbf{u_c})$. We combine (2.3.18), (2.3.19) and (2.3.22) to get

$$\int_0^T \int_{\mathbb{R}^3} (\frac{\partial \psi_t}{\partial t} + < \mathbf{X}, \nabla \psi_t >) d\sigma_t dt = - \iiiint_{\mathcal{K}} \zeta \mathrm{div}(r\mathbf{u_c}) d\lambda dr dz dt$$
$$+ \iiint_{\{z=0\}_* \cup \{z=H\}_* \cup \{r=r_0\}_*} r\zeta \mathbf{u}_t \cdot \mathbf{n}_t d\mathrm{H}^3$$
$$+ \iiint_{\{r=\varsigma(t,\lambda,z)\}} \frac{\varsigma\zeta}{\sqrt{|\nabla_{t,\lambda,z}\varsigma_t|^2+1}} (\partial_t \varsigma_t + \frac{u}{r}\partial_\lambda \varsigma_t + w\partial_z \varsigma_t - v) d\mathrm{H}^3 \tag{2.3.23}$$

Assume (2.3.12) is satisfied. Then the right hand side of (2.3.23) is zero. As we can choose ψ arbitrary in (2.3.23), we conclude that (2.3.13) holds in the distributional sense. Conversely, assume that (2.3.13) holds in the distributional sense. Then, the left hand side of (2.3.23) is zero. Next, if we choose $\zeta = \zeta^1$ arbitrarily in (2.3.23) such that the support of ζ^1 is contained in $int(\mathcal{K})$ then $\zeta^1 = 0$, on $\{z = 0\}_* \cup \{z = H\}_* \cup \{r = r_0\}_* \cup \{r = \varsigma(t, \lambda, z)\}_*$ in light of (2.3.5) and (2.3.23) becomes

$$\iiiint_{int(\mathcal{K})} \zeta^1 div(r\mathbf{u_c}) d\lambda dr dz dt = 0 \tag{2.3.24}$$

As $\partial\mathcal{K}$ is the union of the graphs of 2 dimensional surfaces, $\mathrm{H}^3(\partial\mathcal{K}) = 0$ and so

$$\iiiint_{\mathcal{K}} \zeta^1 div(r\mathbf{u_c}) d\lambda dr dz dt = \iiiint_{int(\mathcal{K})} \zeta^1 div(r\mathbf{u_c}) d\lambda dr dz dt = 0 \tag{2.3.25}$$

for all $\zeta^1 \in C_c^1(\mathcal{K})$. This implies that

$$\mathrm{div}(r\mathbf{u_{c}}_t) = 0 \tag{2.3.26}$$

on Γ_{ς_t} for each t fixed. Thus, (2.3.23) becomes

$$\iiint_{\{z=0\}_* \cup \{z=H\}_* \cup \{r=r_0\}_*} r\zeta \mathbf{u}_t \cdot \mathbf{n}_t d\mathrm{H}^3$$
$$+ \iiint_{\{r=\varsigma(t,\lambda,z)\}_*} \frac{\varsigma\zeta}{\sqrt{|\nabla_{t,\lambda,z}\varsigma_t|^2+1}} (\partial_t \varsigma_t + \frac{u}{r}\partial_\lambda \varsigma_t + w\partial_z \varsigma_t - v) d\mathrm{H}^3 = 0 \tag{2.3.27}$$

Take $\zeta = \zeta_2$ such that the support of ζ_2 is contained in $\mathcal{J}$ (as defined in remark 2.3.1). Thus, ζ_2 vanishes on $\{z = 0\}_* \cup \{z = H\}_* \cup \{r = r_0\}_*$ and so (2.3.27) becomes

$$\iiint_{\{r=\varsigma(t,\lambda,z)\}_*\cap\mathcal{J}} \frac{\varsigma\zeta_2}{\sqrt{|\nabla_{t,\lambda,z}\varsigma_t|^2+1}}(\partial_t\varsigma_t+\frac{u}{r}\partial_\lambda\varsigma_t+w\partial_z\varsigma_t-v)d\mathrm{H}^3=0 \tag{2.3.28}$$

In view of (2.3.9) and (2.3.11), (2.3.28) implies that

$$\iiint_{\{r=\varsigma(t,\lambda,z)\}_*} \frac{\varsigma\zeta_2}{\sqrt{|\nabla_{t,\lambda,z}\varsigma_t|^2+1}}(\partial_t\varsigma_t+\frac{u}{r}\partial_\lambda\varsigma_t+w\partial_z\varsigma_t-v)d\mathrm{H}^3=0 \tag{2.3.29}$$

for all $\zeta_2\in C_c(\mathcal{J})$. Thus,

$$\partial_t\varsigma_t+\frac{u}{r}\partial_\lambda\varsigma_t+w\partial_z\varsigma_t-v=0 \tag{2.3.30}$$

on $\{r=\varsigma_t\}$ for each t fixed. Consequently, (2.3.27) becomes

$$\iiint_{\{z=0\}_*\cup\{z=H\}_*\cup\{r=r_0\}_*} r\zeta\mathbf{u}_t\cdot\mathbf{n}_t d\mathrm{H}^3=0 \tag{2.3.31}$$

for all $\zeta_2\in C_c((0,T)\times\mathbb{R}^2)$. Therefore

$$\mathbf{u}_t\cdot\mathbf{n}_t=0 \tag{2.3.32}$$

on $\{z=0\}\cup\{z=H\}\cup\{r=r_0\}$ for each t fixed. The equations (2.3.26) (2.3.30) (2.3.32) form (2.3.12). □

Let $\Delta\in\mathbb{R}^2_+$ be an open bounded and $\Delta_{r_0}=[0,H]\times[0,1/(2r_0^2))$. We consider the set $\mathcal{S}$ of functions (P,Ψ) such that $P_t^\lambda(s,z):\Delta_{r_0}\longrightarrow\mathbb{R}$ and $\Psi_t^\lambda(\Upsilon,Z):\Delta\longrightarrow\mathbb{R}$ are Legendre transforms of each other for each λ and t fixed. It is well known that if $(P,\Psi)\in\mathcal{S}$ and P_t^λ is strictly convex and differentiable in the interior of its domain then so is Ψ_t^λ and $\nabla_{s,z}P_t^\lambda$ is invertible with inverse $\nabla_{\Upsilon,Z}\Psi_t^\lambda$ in the interior of their domains.

Lemma 2.3.4 *Assume $(P,\Psi)\in\mathcal{S}$ and P_t^λ is strictly convex and differentiable in the interior of its domain.*
If $\lambda\longmapsto P_t(\lambda,s,z)$ is differentiable at $\lambda_0\in(0,2\pi)$ then

$$\partial_\lambda P(\lambda_0,\cdot)=-\partial_\lambda\Psi(\lambda_0,\cdot)\circ\nabla_{s,z}P_{\lambda_0} \tag{2.3.33}$$

Proof: Let $p_0 \in int(\Delta_{r_0})$ and choose q_0 such that

$$\Psi_{\lambda_0}(q_0) + P_{\lambda_0}(p_0) = \langle p_0, q_0 \rangle \tag{2.3.34}$$

Note that as Ψ_t^λ and P_t^λ are Legendre transforms of each other, for all $\lambda \in (0, 2\pi)$ we have

$$\Psi_\lambda(q_0) + P_\lambda(p_0) \geq \langle p_0, q_0 \rangle \tag{2.3.35}$$

We conclude from (2.3.34) and (2.3.35) that

$$\partial_\lambda \Psi(\lambda_0, q_0) = -\partial_\lambda P(\lambda_0, p_0) \tag{2.3.36}$$

By (2.3.34), $q_0 \in \partial . P_{\lambda_0}(p_0) = \{\nabla_{s,z} P_{\lambda_0}(p_0)\}$ as P_{λ_0} is differentiable, and so (2.3.36) becomes

$$\partial_\lambda \Psi(\lambda_0, \cdot) \circ \nabla_{s,z} P_{\lambda_0}(p_0) = -\partial_\lambda P(p_0)$$

$\square$

In the sequel, we define $\mathbf{s} : [0, \infty) \times [0, H] \longrightarrow \Delta_{r_0} := [0, \frac{1}{2r_0^2}) \times [0, H]$ by

$$\mathbf{s}(r, z) = \left(\frac{1}{2}(r_0^{-2} - r^{-2}), z \right)$$

Note that $\mathbf{s}$ is invertible with inverse

$$\mathbf{d}(s, z) = (\mathbf{d}_1(s, z), \mathbf{d}_2(s, z)) = \left(\frac{r_0}{\sqrt{1 - 2r_0^2 s}}, z \right)$$

And so at any point of $(r, z) \in (0, \infty) \times (0, H)$,

$$[\nabla \mathbf{s}]^{-1}(r, z) = [\nabla \mathbf{d}] \circ \mathbf{s}(r, z) = \begin{pmatrix} r^3 & 0 \\ 0 & 1 \end{pmatrix}$$

Let φ and P be functions such that

$$P_t^\lambda \circ \mathbf{s}(r,z) = \varphi_t(\lambda, r, z) + \frac{\Omega^2}{2} r^2 \tag{2.3.37}$$

Remark 2.3.5 *1. As* $\mathbf{s}$ *is injective,* P_t^λ *is strictly convex if and only if* $\varphi_t(\lambda,\cdot) + \Omega^2 \frac{r^2}{2}$ *is strictly convex.*

2. We observe that if $(P,\Psi) \in \mathcal{S}$ *such that* P_t^λ *is strictly convex, differentiable in the interior of its domain and* $\lambda \longmapsto P_t^\lambda(s,z)$ *is differentiable then in view of (2.3.33)*

$$\partial_\lambda \varphi(\lambda,\cdot) = \partial_\lambda P(\lambda,\cdot) \circ \mathbf{s} = -\partial_\lambda \Psi(\lambda,\cdot) \circ \nabla_{s,z} P(\lambda,\cdot) \circ \mathbf{s} \tag{2.3.38}$$

Remark 2.3.6 *Assume* φ *is smooth and let* u, θ' *be functions such that*

$$\frac{u^2}{r} + 2\Omega u = \frac{\partial \varphi}{\partial r} \qquad g\frac{\theta'}{\theta_0} = \frac{\partial \varphi}{\partial z} \tag{2.3.39}$$

Here θ_0*,* g *are positive constants. In view of (2.3.37),* P_t^λ *is differentiable and if* $(P,\Psi) \in \mathcal{S}$ *then*

$$\begin{aligned}
\nabla_{s,z} P_t^\lambda \circ \mathbf{s}(r,z) &= [\nabla \mathbf{s}]^{-T} \left[\nabla_{r,z}\varphi + \Omega(r,0)\right] \\
&= \left(r^3 \frac{\partial \varphi}{\partial r} + \Omega^2 r^2, \frac{\partial \varphi}{\partial z} \right) \\
&= \left((ru + \Omega r^2)^2 + r^3 (\frac{\partial \varphi}{\partial r} - \frac{u^2}{r} - 2u\Omega), \frac{\partial \varphi}{\partial z} \right)
\end{aligned}$$

and so, (2.3.39) is equivalent to

$$[\nabla_{s,z} P_t^\lambda] \circ \mathbf{s} = \left((ru + \Omega r^2)^2, \frac{g}{\theta_0}\theta' \right) \tag{2.3.40}$$

or

$$[\nabla_{s,z} P_t^\lambda] = \left((\mathbf{d}_1 u \circ \mathbf{d} + \Omega \mathbf{d}_1^2)^2, \frac{g}{\theta_0}\theta \circ \mathbf{d} \right) \tag{2.3.41}$$

The first components of (2.3.41) yield

$$\partial_s P_t^\lambda = (\mathbf{d}_1 u \circ \mathbf{d} + \Omega \mathbf{d}_1^2)^2 \tag{2.3.42}$$

We rewrite (2.3.42) as

$$\frac{u \circ \mathbf{d}}{\mathbf{d}_1} = \frac{\sqrt{\partial_s P_t^\lambda}}{\mathbf{d}_1^2} - \Omega$$

or

$$\frac{u \circ \mathbf{d}}{\mathbf{d}_1}(s,z) = \sqrt{\partial_s P_t^\lambda - 2r_0^2 s\sqrt{\partial_s P_t^\lambda}} - \Omega \tag{2.3.43}$$

If in addition Ψ_t^λ (or P_t^λ) is strictly convex then in view of (2.3.43), the equation (2.3.42) is equivalent to

$$\frac{u \circ \mathbf{d}}{\mathbf{d}_1} \circ \nabla_{\Upsilon,Z}\Psi = \sqrt{\Upsilon - 2r_0^2\sqrt{\Upsilon}\partial_\Upsilon \Psi} - \Omega \tag{2.3.44}$$

Note that

$$\mathbf{d} \circ \nabla_{\Upsilon,Z}\Psi_t^\lambda = \left(\frac{r_0}{\sqrt{1 - 2r_0^2 \partial_\Upsilon \Psi}}, \partial_Z \Psi\right)$$

.

Lemma 2.3.7 *F and S are prescribed functions.*

(i) Let $\mathbf{u} = (u, v, w)$ be a smoooth velocity field and θ' φ are smooth real valued functions such that the equations (1.1.2a)-(1.1.2d) are satisfied. Assume that there exists $(P, \Psi) \in \mathcal{S}$ such that (2.3.37) holds. Set $\mathfrak{F}_t(\lambda, \cdot) = (\lambda, \nabla_{s,z}P_t^\lambda \circ \mathbf{s})$. Define the velocity field $\mathbf{X}_t$ by

$$\mathbf{X}_t \circ \bar{\mathfrak{F}}_t = \frac{\partial \bar{\mathfrak{F}}_t}{\partial t} + \langle \mathbf{u_c}, \nabla \bar{\mathfrak{F}}_t\rangle \tag{2.3.45}$$

If P_t^λ is strictly convex then $\mathbf{X}_t$ and Ψ satisfy (1.2.3).

(ii) Conversely, assume $\bar{\mathbf{X}}_t$ and $\bar{\Psi}$ satisfy (1.2.3) with $(\bar{P}, \bar{\Psi}) \in \mathcal{S}$.and $\bar{\Psi}_t^\lambda$ strictly convex. Choose $\bar{\varphi}$ such that (2.3.37) holds for ($\bar{\varphi}$, $\bar{P}$) , a velocity field $\bar{\mathbf{u}} = (\bar{u}, \bar{v}, \bar{w})$ and $\bar{\theta}'$ by setting

$$\bar{\mathfrak{F}}_t(\lambda, \cdot) = (\lambda, \nabla_{s,z}\bar{P}_t^\lambda \circ \mathbf{s}); \qquad \bar{\mathbf{u}}_{\mathbf{c}t} \circ \bar{\mathfrak{F}}_t^{-1} = \frac{\partial \bar{\mathfrak{F}}_t^{-1}}{\partial t} + \langle \bar{\mathbf{X}}_t, \nabla \bar{\mathfrak{F}}_t^{-1}\rangle; \qquad \bar{\theta} = \frac{\theta_0}{g}\frac{\partial \bar{\varphi}}{\partial z} \tag{2.3.46}$$

Then $\bar{\varphi}$, $\bar{\mathbf{u}} = (\bar{u}, \bar{v}, \bar{w})$ and $\bar{\theta}'$ satisfy (1.1.2a)-(1.1.2d).

Proof: (i) Here $\frac{D}{Dt} = \frac{\partial}{\partial t} + \langle \mathbf{u_c}, \nabla_{\lambda,s,z}\rangle$.

We use (2.3.40) to obtain

$$\begin{aligned}\frac{D}{Dt}[\nabla_{s,z}P_t^\lambda \circ \mathbf{s}] &= \left(2(ur+\Omega r^2)\frac{D}{Dt}(ur+\Omega r^2), \frac{g}{\theta_0}\frac{D\theta}{Dt}\right)\\ &= \left(2(\sqrt{\partial_s P_t^\lambda} \circ \mathbf{s})(r\frac{\mathrm{D}u}{\mathrm{D}t} + uv + 2r\Omega v), \frac{g}{\theta_0}\frac{D\theta}{Dt}\right)\end{aligned} \tag{2.3.47}$$

We use (2.3.34) to rewrite (2.3.47) as

$$\begin{aligned}&\frac{D}{Dt}[\nabla_{s,z}P_t^\lambda]\circ \mathbf{s} =\\ &\left(2\sqrt{\partial_s P_t^\lambda}\circ \mathbf{s}\left[r\frac{\mathrm{D}u}{\mathrm{D}t} + uv + 2r\Omega v + \frac{\partial\varphi}{\partial\lambda} + \partial_\lambda \Psi(\lambda,\cdot)\circ\nabla_{s,z}P_t^\lambda \circ \mathbf{s}\right], \frac{g}{\theta_0}\frac{D\theta}{Dt}\right)\end{aligned} \tag{2.3.48}$$

In light of (1.1.2a) and (1.1.2b), (2.3.48) becomes

$$\frac{D}{Dt}[\nabla_{s,z}P_t^\lambda]\circ \mathbf{s} = \left(\sqrt{\partial_s P_t^\lambda}\circ \mathbf{s}\left[F + \partial_\lambda \Psi(\lambda,\cdot)\circ\nabla_{s,z}P_t^\lambda \circ \mathbf{s}\right], \frac{g}{\theta_0}S\right) \tag{2.3.49}$$

and so

$$\begin{aligned}&\frac{D[\nabla_{s,z}P_t^\lambda]\circ \mathbf{s}}{Dt}(\lambda, \mathbf{d}\circ\nabla_{\Upsilon,Z}\Psi_t^\lambda) =\\ &\left(2\sqrt{\Upsilon}\left[F_t(\lambda, \mathbf{d}\circ\nabla_{\Upsilon,Z}\Psi_t^\lambda) + \partial_\lambda\Psi(\lambda,\cdot)\right], \frac{g}{\theta_0}S_t(\lambda, \mathbf{d}\circ\nabla_{\Upsilon,Z}\Psi_t^\lambda)\right)\end{aligned} \tag{2.3.50}$$

We easily check that $\frac{D\lambda}{Dt}\circ \mathbf{d} = \frac{u_t^\lambda \circ \mathbf{d}}{\mathbf{d}_1}$ and exploit (2.3.44) to obtain

$$\frac{D\lambda}{Dt}(\lambda, \mathbf{d}\circ\nabla_{\Upsilon,Z}\Psi_t^\lambda) = \frac{\sqrt{\Upsilon}}{r_0^2} - 2\sqrt{\Upsilon}\partial_\Upsilon\Psi^\lambda(\Upsilon, Z) - \Omega \tag{2.3.51}$$

We note that

$$\frac{D\bar{\mathfrak{F}}_t}{Dt}\circ \bar{\mathfrak{F}}_t^{-1} = \left(\frac{D\lambda}{Dt}(\lambda, \mathbf{d}\circ\nabla_{\Upsilon,Z}\Psi_t^\lambda),\ \frac{D}{Dt}\left[[\nabla_{s,z}P_t^\lambda]\circ \mathbf{s}\right](\lambda, \mathbf{d}\circ\nabla_{\Upsilon,Z}\Psi_t^\lambda)\right) \tag{2.3.52}$$

Also,

$$\mathbf{X}_t = \left(\frac{\partial\bar{\mathfrak{F}}_t}{\partial t} + \langle \mathbf{u_c}, \nabla\bar{\mathfrak{F}}_t\rangle\right)\circ\bar{\mathfrak{F}}_t^{-1} = \frac{D\bar{\mathfrak{F}}_t}{Dt}\circ\bar{\mathfrak{F}}_t^{-1} \tag{2.3.53}$$

We combined (2.3.50), (2.3.51), (2.3.52) and (2.3.53) to obtain

$$\mathbf{X}_t = \left(\frac{\sqrt{\Upsilon}}{r_0^2} - 2\sqrt{\Upsilon}\partial_\Upsilon \Psi - \Omega, 2\sqrt{\Upsilon}\left[F_t(\lambda, \mathbf{d} \circ \nabla_{\Upsilon,Z}\Psi_t^\lambda) + \partial_\lambda \Psi(\lambda, \cdot)\right], \frac{g}{\theta_0} S_t(\lambda, \mathbf{d} \circ \nabla_{\Upsilon,Z}\Psi_t^\lambda) \right) \tag{2.3.54}$$

(ii) Assume now that $\bar{\mathbf{X}}_t = \left(\bar{\mathbf{X}}_t^1, \bar{\mathbf{X}}_t^2, \bar{\mathbf{X}}_t^3\right)$ and $\bar{\Psi}$ satisfy (1.2.3) with $(\bar{P}, \bar{\Psi}) \in \mathcal{S}$. Choose $\bar{\varphi}$ such that (2.3.37) holds, a velocity field $\bar{\mathbf{u}} = (\bar{u}, \bar{v}, \bar{w})$ and $\bar{\theta}'$ as in (2.3.46) The second equation in (2.3.46) implies that (see section 2.3.1)

$$\bar{\mathbf{X}}_t = \left(\frac{\partial \bar{\mathfrak{F}}_t}{\partial t} + \langle \bar{\mathbf{u}}_c, \nabla \bar{\mathfrak{F}}_t \rangle \right) \circ \bar{\mathfrak{F}}_t^{-1} = \frac{D\bar{\mathfrak{F}}_t}{Dt} \circ \bar{\mathfrak{F}}_t^{-1} \tag{2.3.55}$$

Here $\frac{D}{Dt} = \frac{\partial}{\partial t} + \langle \bar{\mathbf{u}}_c, \nabla_{\lambda,s,z} \rangle$. Exploiting the first components equality in (2.3.55)

$$\frac{\sqrt{\Upsilon}}{r_0^2} - 2\sqrt{\Upsilon}\partial_\Upsilon \bar{\Psi}^\lambda(\Upsilon, Z) - \Omega = \mathbf{X}_t^1 = \frac{D\lambda}{Dt} \circ \mathbf{d}(\lambda, \mathbf{d} \circ \nabla_{\Upsilon,Z}\bar{\Psi}_t^\lambda) = \frac{\bar{u}_t^\lambda \circ \mathbf{d}}{\mathbf{d}_1}(\lambda, \mathbf{d} \circ \nabla_{\Upsilon,Z}\bar{\Psi}_t^\lambda) \tag{2.3.56}$$

By remark 2.3.6 , (2.3.56) is equivalent to

$$\partial_s \bar{P}_t^\lambda \circ \mathbf{s}(r, z) = (r\bar{u} + \Omega r^2)^2 \tag{2.3.57}$$

Note that as (2.3.37) holds,

$$\bar{\theta}' = \frac{\theta_0}{g} \frac{\partial \bar{\varphi}}{\partial z} = \frac{\theta_0}{g} \frac{\partial \bar{P}_t^\lambda}{\partial z} \circ \mathbf{s} \tag{2.3.58}$$

Combining (2.3.57) and (2.3.58) we obtain

$$[\nabla_{s,z} \bar{P}_t^\lambda] \circ \mathbf{s} = \left((r\bar{u} + \Omega r^2)^2, \frac{g}{\theta_0}\bar{\theta} \right) \tag{2.3.59}$$

and so by remark 2.3.6 again u, θ' and φ solve (1.1.2c) and (1.1.2d)

We compose the equality in (2.3.55) by $\mathfrak{F}$ and exploit the equality in the second and third components to obtain that

$$\left(\sqrt{\partial_s \bar{P}_t^\lambda} \circ \mathbf{s}\left[F_t + \partial_\lambda \bar{\Psi}(\lambda,\cdot) \circ \nabla_{s,z} \bar{P}_t^\lambda \circ \mathbf{s}\right], \frac{g}{\theta_0} S_t\right)$$
$$= \left(\bar{\mathbf{X}}_t^2(\lambda, \nabla_{s,z} P_t^\lambda \circ \mathbf{s}), \bar{\mathbf{X}}_t^3(\lambda, \nabla_{s,z} P_t^\lambda \circ \mathbf{s})\right) \tag{2.3.60}$$
$$= \frac{D}{Dt}[\nabla_{s,z} \bar{P}_t^\lambda \circ \mathbf{s}]$$

In view of (2.3.48), the equation (2.3.60) implies that

$$\frac{\mathrm{D}\bar{u}}{\mathrm{D}t} + \frac{\bar{u}\bar{v}}{r} + 2\Omega\bar{v} + \frac{1}{r}\frac{\partial\bar{\varphi}}{\partial\lambda} = \frac{1}{r}F_t \qquad \frac{\mathrm{D}\bar{\theta}'}{\mathrm{D}t} = S_t$$

that is, $\bar{\mathbf{u}}, \bar{\varphi}, \bar{\theta}'$ solve (1.1.2a) and (1.1.2b).

□

Theorem 2.3.8 *Assume for the sake of simplicity that $r_0 < \varsigma$. Assume $\mathbf{u} = (u, v, w)$ θ' φ and ς are smooth and solve (1.0.2) and (1.0.3). We assume in addition that*

$$\theta' \geq 0 \qquad \textit{and} \qquad \nabla^2_{r,z}\left(\frac{\Omega^2 r^2}{2} + \varphi_t^\lambda(r,z)\right) > 0. \tag{2.3.61}$$

Let $(P, \Psi) \in \mathcal{S}$ such that (2.3.37) holds. Define $\mathbf{X}_t$ as in (2.3.45). Set $\mathfrak{F}_t(\lambda,\cdot) = (\lambda, \nabla_{s,z} P_t^\lambda \circ \mathbf{s})$. If $\mathfrak{F}_t$ pushes forwards $r\chi_{\Gamma_{\varsigma_t}}$ onto σ_t then σ_t and $\mathbf{X}_t$ solve the continuity equation in (1.2.2) and $\mathbf{X}_t$ and Ψ satisfy (1.2.3).
Conversely, assume $\bar{\mathbf{X}}_t$ and $\bar{\Psi}$ satisfy (1.2.3) with $(\bar{P}, \bar{\Psi}) \in \mathcal{S}$ and $\bar{\Psi}_t^\lambda$ strictly convex. Assume $\bar{\sigma}_t$ is absolutely continuous with respect to Lebesgue, such that $\bar{\sigma}_t$ and $\mathbf{X}_t$ solve the continuity equation in (1.2.2). Assume that there exists a function $\bar{\varsigma}$ such that $\mathbf{d} \circ \nabla \bar{\Psi}_t^\lambda$ pushes forward $\bar{\sigma}_t^\lambda$ to $r\chi_{\Gamma_{\bar{\varsigma}_t^\lambda}}$. Choose $\bar{\varphi}$ such that (2.3.37) holds for $(\bar{\varphi}, \bar{P})$, a velocity field $\bar{\mathbf{u}} = (\bar{u}, \bar{v}, \bar{w})$ and $\bar{\theta}'$ by setting

$$\bar{\mathfrak{F}}_t(\lambda,\cdot) = (\lambda, \nabla_{s,z} \bar{P}_t^\lambda \circ \mathbf{s}); \qquad \bar{\mathbf{u}}_{\mathbf{c}t} \circ \bar{\mathfrak{F}}_t^{-1} = \frac{\partial \bar{\mathfrak{F}}_t^{-1}}{\partial t} + \langle \bar{\mathbf{X}}_t, \nabla \bar{\mathfrak{F}}_t^{-1} \rangle; \qquad \bar{\theta}' = \frac{\theta_0}{g}\frac{\partial \bar{\varphi}}{\partial z} \tag{2.3.62}$$

Then $\bar{\varphi}$, $\bar{\mathbf{u}} = (\bar{u}, \bar{v}, \bar{w})$ and $\bar{\theta}'$ solve (1.0.2) and (1.0.3).

Proof: As φ satisfies the second equation in (2.3.61), P_t^λ is strictly convex. Therefore, $\mathfrak{F}_t$ is invertible for each t fixed. Thus, the conditions in (2.3.15) are satisfied. As $\mathbf{u} = (u, v, w)$ and φ solve (1.1.2e) and (1.0.3), by lemma 2.3.3, σ_t and $\mathbf{X}_t$ solve the continuity equation in (1.2.2). Lemma 2.3.7 (i) ensures that (1.2.3) holds. we conclude that (i) is proved.

The fact that $\mathbf{d} \circ \nabla \bar{\Psi}_t^\lambda$ pushes forward $\bar{\sigma}_t^\lambda$ to $r\chi_{\Gamma_{\bar{\varsigma}_t^\lambda}}$ implies that $\nabla \bar{P}_t^\lambda \circ \mathbf{s}$ pushes forward $r\chi_{\Gamma_{\bar{\varsigma}_t^\lambda}}$ to $\bar{\sigma}_t^\lambda$. Let $G : [0, 2\pi] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be continuous.

$$
\begin{aligned}
\int_{\Gamma_\varsigma} G \circ \bar{\mathfrak{F}}_t(\lambda, r, z) r dr dz d\lambda &= \int_0^{2\pi} d\lambda \int_{\Gamma_{\varsigma^\lambda}} G(\lambda, \cdot) \circ \nabla_{s,z} \bar{P}_t^\lambda \circ \mathbf{s}(r, z)\; r dr dz \\
&= \int_0^{2\pi} d\lambda \int_{\Gamma_{\varsigma^\lambda}} G(\lambda, \cdot) \sigma^\lambda d\Upsilon dZ \\
&= \int_{\Gamma_{\varsigma^\lambda}} G \sigma d\Upsilon dZ d\lambda
\end{aligned}
$$

As G is an arbitrary continuous function, we conclude that $\bar{\mathfrak{F}}_t$ pushes $r\chi_{\Gamma_{\bar{\varsigma}_t}}$ to $\bar{\sigma}_t$. This, along with the second equation in (2.3.62) fulfilled the conditions in (2.3.15). Thus, as $\bar{\sigma}_t$ and $\bar{\mathbf{X}}_t$ solve the continuity equation in (1.2.2) by lemma 2.3.3 again, $\bar{\mathbf{u}} = (\bar{u}, \bar{v}, \bar{w})$ and $\bar{\varphi}$ solve (1.1.2e) and (1.0.3). Lemma 2.3.7 (ii) ensures that $\bar{\varphi}$, $\bar{\mathbf{u}} = (\bar{u}, \bar{v}, \bar{w})$ and $\bar{\theta}'$ solve (1.1.2a-1.1.2d).

2.4 Connection between the Almost axisymmetric Flows with Forcing Term and Monge-Ampere equation. Program for solving the Almost Axisymmetric Flows with Forcing Terms.

We recall that solving the Monge-Ampere equation in (1.4.1) comes down essentially to solving the variational problem (1.4.4) and its dual formulation (1.4.5). As we will soon show, when $\{\bar{\sigma}_t\}_{t\in(0,T)}$ is absolutely continuous with respect to Lebesgue, there exist $(\bar{\Psi}, \bar{P}, \bar{h})$ such that $(\bar{\Psi}_t^\lambda, \bar{P}_t^\lambda)$ Legendre transforms of each other, Lipschitz, maximizer in (1.4.5) and h_t^λ minimizer in (1.4.4) such that

$$\nabla_{\Upsilon,Z}\bar{\Psi}^\lambda_{t\#}\bar{\sigma}^\lambda_t = e(s)\chi_{D_{\bar{h}^\lambda_t}}$$

$e(s)$ and D_h are defined in section 1.4. That is,

$$\mathbf{d}\circ\nabla_{\Upsilon,Z}\bar{\Psi}^\lambda_{t\#}\bar{\sigma}^\lambda_t = r\chi_{\Gamma_{\bar{\varsigma}^\lambda_t}} \qquad \bar{\varsigma}^\lambda_t = \frac{r_0}{\sqrt{1-2r_0^2h^\lambda_t}}$$

$\mathbf{d}$ is defined as in the previous section. Moreover,

$$\nabla_{\Upsilon,Z}\Psi^\lambda_t\circ\nabla_{s,z}P^\lambda_t = \mathbf{id} \quad e(s)\chi_{D_{\bar{h}^\lambda_t}}(s,z)\mathcal{L}^2 \quad a.e \quad \text{and} \quad \nabla_{s,z}P^\lambda_t\circ\nabla_{\Upsilon,Z}\Psi^\lambda_t = \mathbf{id} \quad a.e \quad \bar{\sigma}^\lambda_t$$

Here is the program we plan to execute in order to solve the Almost axisymmetric Flows with Forcing Term.

(a) Note that if $\bar{\Psi}$ has enough regularity with respect to λ then the corresponding velocity field $\bar{\mathbf{X}}_{\bar{\sigma}}$ in (1.2.3) exists ($\bar{\sigma}_t$ almost everywhere) and is well defined. So, ensuring the existence of the velocity field brings us back to the regularity of the solution of the Monge-Ampere equation with respect to a parameter discussed in section 1.5.

(b) The next challenge will consist of finding a class of initial data σ_0 and appropriate conditions on the velocity fields so that the solutions to (1.2.2) stay absolutely continuous with respect to Lebesgue and enough regularity of $\bar{\Psi}$ with respect to λ is maintained as time evolves in a discrete scheme.

(c) In the case, where the forcing terms $F = S = 0$ in (1.0.2), we obtain the Almost Axisymmetric Flows. These Flows are expected to have an Hamiltonian structure.

(d) Theorem 2.3.8 connects the Almost axisymmetric Flows with Forcing Terms to the Monge-Ampere equations in the following way: it shows that if we had had enough regularity on $(\bar{\Psi}_t, \bar{P}_t, \bar{h}_t)$ solving the Monge Ampere equation (1.4.1) for $\bar{\sigma}_t$ given for each t fixed and if $\bar{\sigma}_t$ and $\bar{\mathbf{X}}_{\bar{\sigma}_t}$ had solved the continuity equation in (1.2.2) then we would have obtained a solution to the Almost Axisymmetric Flows with Forcing terms (1.0.2) and (1.0.3).

Thus, the regularity of $(\bar{\Psi}, \bar{P}, \bar{h})$ becomes crucial in solving (1.0.2) and (1.0.3).

CHAPTER III

MINIMIZATION PROBLEM

In this chapter, we propose two different approaches to the minimization problem (1.4.4). The first approach draws on the techniques of direct method of Calculus of variations to prove the existence of a minimizer. However, we were unable to obtain uniqueness, as the functional $\bar{I}[\sigma]$ in not strictly convex (or even convex) with respect to any metric we can think of. The second approach relies on duality techniques. We invent a dual problem that provides the existence and uniqueness result.

Throughout this Chapter, R_0, r_0, H are positive and prescribed. We set

$$\Delta_{r_0} := [0, H] \times [0, \frac{1}{2r_0^2}).$$

and

$$e(s) = \frac{r_0^4}{(1 - 2sr_0^2)^2} \qquad 0 \le 2r_0^2 s < 1.$$

We set

$$p = (r, z) \qquad \text{and} \qquad q = (\Upsilon, Z)$$

3.1 Continuity properties of $\bar{I}[\sigma]$ and compactness of the set of admissible functions.

We set f and g to be

$$f(p) := \frac{\Omega^2}{2}\sqrt{e(s)} - \frac{|p|^2}{2} \text{ and } g(q) := \frac{\Upsilon}{2r_0^2} - \Omega\sqrt{\Upsilon} - \frac{|q|^2}{2}$$

Here,

$$p = (r, z) \in \Delta_{r_0} \qquad \text{and} \qquad q = (\Upsilon, Z) \in \mathbb{R}^2_+$$

Let $\sigma \in \mathcal{P}([0,R_0]^2)$. To any $h:[0,H] \longrightarrow [0,1/(2r_0^2))$, we associate the functional

$$\bar{I}[\sigma](h) = \frac{1}{2}W_2^2(\sigma, e(s)\chi_{D_h}) + \int_{\mathbb{R}^2} f(p)e(s)\chi_{D_h}(dq) + \int_{\mathbb{R}^2} g(q)\sigma(dq). \tag{3.1.1}$$

with

$$D_h = \{(s,z) : 0 \le z \le H,\ 0 \le s \le h(z)\ \}$$

We define

$$\mathcal{H}_{dom} = \left\{ h:[0,H] \longrightarrow [0,1/(2r_0^2))\ |\ \int_{\mathbb{R}^2} e(s)\chi_{D_h} dsdz = 1\ \right\}$$

which can also be rewritten as

$$\mathcal{H}_{dom} = \left\{ h:[0,H] \mapsto [0,\frac{1}{2r_0^2})|:\ ||e^{1/2}\circ h||_{L^1(0,H)} = 2 + Hr_0^2 \right\} \tag{3.1.2}$$

If $h \in \mathcal{H}_{dom}$ then $e(s)\chi_{D_h}$ and σ have finite second moments so that $W_2^2(\sigma, e(s)\chi_{D_h})$ is finite.

We note that $W_2^2(\sigma, e(s)\chi_{D_h})$ is finite if and only if $h \in \mathcal{H}_{dom}$.

Remark 3.1.1 *Note that $\frac{d}{dt}e(s) = e(s)^{\frac{3}{2}}$ and so*

$$\int_{D_h} \sqrt{e(s)}e(s)dsdz = \int_0^H \int_0^{h(z)} e(s)^{\frac{3}{2}} dsdz = \frac{1}{4}\int_0^H e\circ h(z) - e(0)dz \tag{3.1.3}$$

If $h \in \mathcal{H}_{dom}$ then

$$\int_{D_h} \frac{|p|^2}{2} e(s)dsdz \le \max_{p\in\bar{\Delta}_{r_0}} \frac{|p|^2}{2} < \infty$$

and in view of (3.1.3)

$$\left| \int_{\mathbb{R}^2} f(p)e(s)\chi_{D_h}(dq) \right| < \infty$$

if and only if $||e\circ h||_{L^1[0,H]} < \infty$.

As $\sigma \in \mathcal{P}([0,R_0]^2)$ and g is continuous,

$$\int_{\mathbb{R}^2} g(q)\sigma(dq) =: C_0(R_0) < \infty$$

and so $\bar{I}[\sigma](h)$ is finite if and only if $\mathcal{H}_{dom} \cap \{h : ||e\circ h||_{L^1[0,H]} < \infty\}$

We easily check that the constant function $h_{00}(z) = \frac{1}{r_0^2(2+Hr_0^2)}$ belongs to the set $\mathcal{H}_{dom} \cap$

$\{h : ||e \circ h||_{L^1[0,H]} < \infty\}$.

Set

$$C_1(R_0, r_0, H) := \max_{(p,q)\in\bar{\Delta}_{r_0}\times[0,R_0]^2} \left(\frac{|p-q|^2}{2}\right)$$

If $h \in \mathcal{H}_{dom}$ *then*

$$W_2^2(\sigma, e(s)\chi_{D_h}) \leq 2C_1(R_0, r_0, H)$$

Lemma 3.1.2 *Let* $\sigma \in \mathcal{P}([0, R_0]^2)$. *Then there exist constants* $c_0 = c_0(R_0, r_0, H)$ *and* $\bar{c}_0 = \bar{c}_0(R_0, r_0, H)$ *independent of* σ *such that*

$$\frac{\Omega^2}{8}||e \circ h||_{L^1[0,H]} + \bar{c}_0 \geq I[\sigma](h) \geq \frac{\Omega^2}{8}||e \circ h||_{L^1[0,H]} + c_0$$

for all $h \in \mathcal{H}_{dom}$.

Proof: Set

$$c_1(r_0, H) := \max_{(s,z)\in\bar{\Delta}_{r_0}} \left(\frac{s^2+z^2}{2}\right)$$

We note that

$$\frac{\Omega^2}{2}\sqrt{e}(s) \geq f(s,z) = \frac{\Omega^2}{2}\sqrt{e}(s) - \frac{|p|^2}{2} \geq \frac{\Omega^2}{2}\sqrt{e}(s) - c_1$$

In view of (3.1.3), for any $h \in \mathcal{H}_{dom}$,

$$\begin{aligned}
\bar{I}[\sigma](h) \geq &\int f(p)e(s)\chi_{D_h}(dq) + \int_{\mathbb{R}^2} g(q)\sigma(dq) \\
=&\frac{\Omega^2}{2}\int_{D_h}\sqrt{e}(s)e(s)dsdz - c_1\int_{D_h} e(s)dsdz + C_0(R_0). \\
=&\frac{\Omega^2}{8}\int_0^H e\circ h(z) - e(0)dz - c_1(r_0,H) + C_0(R_0) \qquad (3.1.4)\\
=&\frac{\Omega^2}{8}||e\circ h||_{L^1[0,H]} - \frac{H\Omega^2}{8}e(0) - c_1(r_0,H) + c_2(R_0) \\
=&\frac{\Omega^2}{8}||e\circ h||_{L^1[0,H]} + c_0
\end{aligned}$$

On the other hand, for any $h \in \mathcal{H}_{dom}$,

$$\begin{aligned}
\bar{I}[\sigma](h) \leq & C_1(R_0, r_0, H) + \int f(p)e(s)\chi_{D_h}(dq) + \int_{\mathbb{R}^2} g(q)\sigma(dq) \\
= & C_1(R_0, r_0, H) + \frac{\Omega^2}{2}\int_{D_h} \sqrt{e}(s)e(s)dsdz + C_0(R_0). \\
= & \frac{\Omega^2}{8}\int_0^H e \circ h(z) - e(0)dz + C_1(R_0, r_0, H) + C(R_0) \\
= & \frac{\Omega^2}{8}||e \circ h||_{L^1[0,H]} - \frac{H\Omega^2}{8}e(0) + C_1(R_0, r_0, H) + c_2(R_0) \\
= & \frac{\Omega^2}{8}||e \circ h||_{L^1[0,H]} + \bar{c}_0
\end{aligned} \tag{3.1.5}$$

The result follows directly.

Lemma 3.1.3 *Let $C > 0$. Assume $\{h_n\}_{n=1}^{\infty}$ converges h almost everywhere with respect to Lebesgue and $\{h_n\}_{n=1}^{\infty}, h \in \mathcal{H}_{dom}$. Then*

$$e(s)\chi_{D_{h_n}} \rightharpoonup e(s)\chi_{D_h} \text{ weakly*}$$

Moreover, if $||e \circ h_n||_{L^1[0,H]} \leq C$ then

$$e(s)\chi_{D_{h_n}} \rightharpoonup e(s)\chi_{D_h} \text{ narrowly}$$

Proof: Let $\phi \in C_c(\Delta_{r_0})$.

$$\int_{D_{h_n}} e(s)\phi dsdz = \int_0^H dz \int_0^{h_n(z)} e(s)\phi(s,z)ds = \int_0^H A(h_n(z), z)dz$$

where

$$A(t,z) = \int_0^t \phi(s,z)e(s)ds$$

Let $0 < M < \frac{1}{2r_0^2}$ such that $\phi(s,z) = 0$ whenever $s > M$. Then

$$A(t,z) = \int_0^{min(M,t)} e(s)\phi(s,z)ds.$$

Therefore,

$$|A(t,z)| \leq Me(M)||\phi||_\infty \tag{3.1.6}$$

for all $0 \leq t < \frac{1}{2r_0^2}$ and $z \in [0, H]$. Since $h_n \longrightarrow h$ a.e and $A(., z)$ is continuous

$$A(h_n(z), z) \longrightarrow A(h(z), z) \quad a.e \tag{3.1.7}$$

Using (3.1.6) and (3.1.7) we apply the Lebesgue dominated convergence theorem to get

$$\int_{\mathbb{R}^2} \phi e(s)\chi_{D_{h_n}} dp = \int_0^H A(h_n, z)dz \longrightarrow \int_0^H A(h, z)dz = \int_{\mathbb{R}^2} \phi e(s)\chi_{D_h} dp$$

Hence $\left\{e(s)\chi_{D_{h_n}}\right\}_{n=1}^{\infty}$ converges weakly* to $e(s)\chi_{D_h}$.

By (3.1.3),

$$\int_{\Delta_{r_0}} \sqrt{e(s)}e(s)\chi_{D_{h_n}} dp = \frac{1}{4}||e \circ h_n||_{L^1[0,H]} - \frac{H}{4}e(0)$$

We extract from $\{h_n\}_{n=1}^{\infty}$ a subsequence that we still denote by $\{h_n\}_{n=1}^{\infty}$.

Since $||e \circ h_n||_{L^1[0,H]} \leq C$, we have

$$\sup_n \int_{\mathbb{R}^2} \sqrt{e(s)}e(s)\chi_{D_{h_n}}(s, z)dsdz < \infty \tag{3.1.8}$$

Note that for any $c > 0$

$$\left\{(s, z) \in \Delta_{r_0} : \sqrt{e(s)} \leq c\right\} = \left[0, \frac{1}{2r_0^2} - \frac{1}{2c}\right] \times [0, H]$$

Therefore the c-sublevels of $(s, z) \mapsto \sqrt{e(s)}$ are compact in Δ_{r_0}. Hence, there exists a subsequence $\{h_{n_k}\}_{k=1}^{\infty}$ of $\{h_n\}_{n=1}^{\infty}$ such that $\left\{e(s)\chi_{D_{h_{n_k}}}\right\}_{k=1}^{\infty}$ converges narrowly (and therefore Weakly*) to some probability measure μ on Δ_{r_0} (see [[1], Remark 5.1.5]). As weak* convergence is metrizable, the limit is unique and $\mu = e(s)\chi_{D_h}$ as obtained in the first part of the proof. We conclude that the whole sequence $\left\{e(s)\chi_{D_{h_n}}\right\}_{n=1}^{\infty}$ converges narrowly to $e(s)\chi_{D_h}$.

Corollary 3.1.4 *Let $c > 0$. Assume $\{\sigma_n\}_{n=1}^{\infty}, \quad \sigma \in \mathcal{P}([0, R_0]^2)$ such that $\{\sigma_n\}_{n=1}^{\infty}$ converges narrowly to σ. Then $\bar{I}[\sigma_n](h)$ converges to $\bar{I}[\sigma](h)$ for all $h \in \mathcal{H}_{dom}$. Assume in addition that $\{h_n\}_{n=1}^{\infty}$, h is a sequence of functions in $\mathcal{H}_{dom}$ such that $\{h_n\}_{n=1}^{\infty}$*

converges to h almost everywhere and $\bar{I}[\sigma_n](h_n) \leq c$ for all $n \geq 1$. Then $\bar{I}[\sigma_n](h_n)$ converges to $\bar{I}[\sigma](h)$.

Proof: (a) Let $h \in \mathcal{H}_{dom}$. As $\text{spt}\sigma_n \subset [0, R_0]^2$, we use the continuity of $W_2\left(\cdot, e(s)\chi_{D_h}\right)$ and the fact that $g \in C([0, R_0]^2)$ to obtain that $\bar{I}[\sigma_n](h)$ converges to $\bar{I}[\sigma](h)$.
(b) Since $\bar{I}[\sigma_n](h_n) \leq c$, lemma 3.1.2 implies that $||e \circ h_n||_{L^1[0,H]} \leq C$

(C is a constant independent of n) and in view of lemma 3.1.3, $\left\{e(s)\chi_{D_{h_n}}\right\}_{n=1}^{\infty}$ converges narrowly to $e(s)\chi_{D_h}$. As $D_{h_n} \subset \Delta_{r_0}$ and $\text{spt}\sigma_n, \subset [0, R_0]^2$, we use the continuity of W_2, the facts that $f \in C(\bar{\Delta}_{r_0})$ and $g \in C([0, R_0]^2)$ to conclude that $\bar{I}[\sigma_n](h)$ converges to $\bar{I}[\sigma](h)$. $\square$

Lemma 3.1.5 *Let $c_0 > 0$. Assume $\{h_n\}_{n=1}^{\infty} \subset \mathcal{H}_{dom}$ and $h : [0, H] \longmapsto [0, \frac{1}{2r_0^2})$ such that $\{h_n\}_{n=1}^{\infty}$ converges a.e to h and $||e \circ h_n||_{L^1[0,H]} < c_0$. Then $h \in \mathcal{H}_{dom}$.*

Proof: Let $\varepsilon > 0$, since $h_n \longrightarrow h$ a.e and $\sqrt{e}$ is continuous, $\sqrt{e} \circ h_n \longrightarrow \sqrt{e} \circ h$ a.e. By Egorov's theorem, there exists a measurable subset A of $[0, H]$ such that $\sqrt{e} \circ h_n \longrightarrow \sqrt{e} \circ h$ uniformly on $[0, H] \setminus A$, and $|A| < \varepsilon$ (here for convenience, $|\cdot|$ denotes the one-dimensional Lebesgue measure). So

$$\lim_{n\to\infty} \int_{[0,H]\setminus A} \sqrt{e} \circ h_n dz = \int_{[0,H]\setminus A} \sqrt{e} \circ h dz \tag{3.1.9}$$

Jensen's inequality implies that

$$\left(\frac{1}{|A|}\int_A \sqrt{e} \circ h_n dz\right)^2 \leq \frac{1}{|A|}\int_A e \circ h_n dz$$

Therefore

$$\int_A \sqrt{e} \circ h_n dz \leq \sqrt{|A|}\sqrt{\int_A e \circ h_n dz} \leq \sqrt{c_0}\sqrt{\varepsilon} \tag{3.1.10}$$

Recall that $g \in \mathcal{H}_{dom}$ is equivalent to $||\sqrt{e} \circ g||_{L[0,H]} = 2 + Hr_0^2$. We exploit (3.1.9) and (3.1.10) to obtain

$$
\begin{aligned}
\int_0^H \sqrt{e} \circ h dz &\geq \int_{[0,H]\setminus A} \sqrt{e} \circ h dz \\
&= \lim_{n\to\infty} \left(\int_{[0,H]} \sqrt{e} \circ h_n dz - \int_A \sqrt{e} \circ h_n dz \right) \\
&\geq 2 + H r_0^2 - \sqrt{c_0}\sqrt{\varepsilon}
\end{aligned} \tag{3.1.11}
$$

As $\varepsilon >$ is arbitrary, (3.1.11) implies that

$$
\int_0^H \sqrt{e} \circ h dz \geq 2 + H r_0^2 \tag{3.1.12}
$$

By virtue of Fatou's lemma, we have

$$
\int_0^H \sqrt{e} \circ h dz \leq \int_0^H \liminf_{n\to\infty} \sqrt{e} \circ h_n dz \leq \liminf_{n\to\infty} \int_0^H \sqrt{e} \circ h_n dz = 2 + H r_0^2 \tag{3.1.13}
$$

We combine (3.1.12) and (3.1.13) to obtain

$$
\int_0^H \sqrt{e} \circ h dz = 2 + H r_0^2
$$

□

3.2 Minimization using Direct Methods of the Calculus of Variations

In this section, π_1 is the first projection in $\mathbb{R}^2$ defined by $\pi_1(a,b) = a$ for all $(a,b) \in \mathbb{R}^2$. Recall that

$$
\mathcal{H}_{dom} := \left\{ h : [0,H] \longrightarrow [0, 1/(2r_0^2)) \mid \int_{\mathbb{R}^2} e(s)\chi_{D_h} ds dz = 1 \right\}
$$

Let $\sigma \in \mathcal{P}([0,R_0]^2)$. Recall

$$
\bar{I}[\sigma](h) = \frac{1}{2} W_2^2(\sigma, e(s)\chi_{D_h}) + \int_{\mathbb{R}^2} f(p)\sigma(dp) + \int g(q) e(s)\chi_{D_h}(dq).
$$

with

$$
f(p) := \frac{\Omega^2 r_0^2}{2(1-2sr_0^2)} - \frac{|p|^2}{2} \text{ and } g(q) := \frac{\Upsilon}{2r_0^2} - \Omega\sqrt{\Upsilon} - \frac{|q|^2}{2}
$$

Let $h \in \mathcal{H}_{dom}$. Then,

$$\frac{1}{2}W_2^2(\sigma, e(s)\chi_{D_h}) = \inf_{\gamma\in\Gamma(\sigma,e(s)\chi_{D_h})} \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|p-q|^2}{2}\gamma(dp,dq)$$
$$= \inf_{\gamma\in\Gamma(\sigma,e(s)\chi_{D_h})} \int_{\mathbb{R}^2\times\mathbb{R}^2} -<q,p>\gamma(dp,dq)$$
$$+\int_{\mathbb{R}^2} \frac{|p|^2}{2}\chi_{D_h}(p)dp + \int_{\mathbb{R}^2}\frac{|q|^2}{2}\sigma(dq)$$

we check easily that

$$\bar{I}[\sigma](h) = \inf_{\gamma\in\Gamma(\sigma,e(s)\chi_{D_h})} \int_{\mathbb{R}^2\times\mathbb{R}^2} c(p,q)\gamma(dp,dq) + \int_{\mathbb{R}^2} \frac{\Upsilon}{2r_0^2} - \Omega\sqrt{\Upsilon}\sigma(dq). \tag{3.2.1}$$

where

$$c(p,q) = -<q,p> + \frac{\Omega^2}{2}e^{\frac{1}{2}}\circ\pi_1(p)$$

If σ is absolutely continuous with respect to Lebesgue and h is such that

$$h \in \mathcal{H}_{dom} \qquad \text{and} \qquad \bar{I}[\sigma](h) < \infty \tag{3.2.2}$$

then (3.2.1) admits a unique minimizer $\bar{\gamma}$. Moreover, there exist a $e(s)\chi_{D_h}\mathcal{L}^2-$measurable function T that pushes forward $e(s)\chi_{D_h}\mathcal{L}^2$ to σ and a $\sigma-$measurable function S that pushes forward σ to $e(s)\chi_{D_h}\mathcal{L}^2$ such that

$$\bar{\gamma} = (\mathbf{id}\times S)_{\#}\sigma = (T\times\mathbf{id})_{\#}e(s)\chi_{D_h}(s,z) \tag{3.2.3}$$

In thise case

$$T\circ S(q) = q \;\; \sigma \; a.e \qquad S\circ T(s,z) = (s,z) \;\; e(s)\chi_{D_h}(s,z)\mathcal{L}^2 \; a.e. \tag{3.2.4}$$

(see [46] Page 67) and

$$\bar{I}[\sigma](h) = \int_{\mathbb{R}^2\times\mathbb{R}^2} c(S(q),q)\sigma(dq) + \int_{\mathbb{R}^2}\frac{\Upsilon}{2r_0^2} - \Omega\sqrt{\Upsilon}\sigma(dq). \tag{3.2.5}$$

Remark 3.2.1 *Assume σ is absolutely continuous with respect to Lebesgue and h satisfies (3.2.2) and choose S and T as provided by (3.2.3).*

(a) Set

$$\mathcal{M}_0 = \{T\circ S = \mathbf{id}_{\mathbb{R}^2}\} \qquad \text{and} \qquad \mathcal{N}_0 = \{S\circ T = \mathbf{id}_{\mathbb{R}^2}\}$$

By (3.2.4), $\mathcal{M}_0$ has full measure with respect to σ and $\mathcal{N}_0$ has full measure with respect to $e(s)\chi_{D_h}(s,z)\mathcal{L}^2$. We notice that S is injective on $\mathcal{M}_0$. We next consider the sets

$$\mathcal{M}_\sigma = \mathcal{M}_0 \cap spt(\sigma) \qquad \textit{and} \qquad \mathcal{N}_h = \mathcal{N}_0 \cap D_h$$

As $\mathcal{N}_0$ and $\mathcal{M}_0$ have full measures with respect to $e(s)\chi_{D_h}\mathcal{L}^2$ and σ respectively, by construction $\mathcal{N}_\sigma$ and $\mathcal{M}_\sigma$ have full measure with respect to $e(s)\chi_{D_h}\mathcal{L}^2$ and σ respectively. We also note that since S pushes σ onto $e(s)\chi_{D_h}(s,z)\mathcal{L}^2$, and $\mathcal{N}_h$ has full measure with respect to $e(s)\chi_{D_h}\mathcal{L}^2$, $S^{-1}(\mathcal{N}_h)$ has full measure as well with respect to σ.

Define

$$\mathcal{C}(h,S,\sigma) := S^{-1}(\mathcal{N}_h) \cap \mathcal{M}_\sigma \qquad \textit{and} \qquad \mathcal{E}(h,S,\sigma) := \mathcal{N}_h \cap S(\mathcal{M}_\sigma)$$

We easily check that $\mathcal{C}(h,S,\sigma)$ and $\mathcal{E}(h,S,\sigma)$ have full measures respectively with respect to σ and $e(s)\chi_{D_h}(s,z)$. As S is injective on $\mathcal{M}_0$ and $\mathcal{C}(h,S,\sigma) \subset \mathcal{M}_0$, we have

$$S\left(\mathcal{C}(h,S,\sigma)\right) = S\left(S^{-1}(\mathcal{N}_h) \cap \mathcal{M}_\sigma\right) = \mathcal{N}_h \cap S\left(\mathcal{M}_\sigma\right) = \mathcal{E}(h,S,\sigma) \tag{3.2.6}$$

and so, S bijectively sends $\mathcal{E}(h,S,\sigma)$ to $\mathcal{C}(h,S,\sigma)$

(b) If $\mathbf{v} \in \mathbb{R}^{\mathbf{2}}$ such that $\pi_1(\mathbf{v}) = 0$ and $U_0 \subset \mathbb{R}^2$ such that $e(s)\chi_{U_0}(s,z)\mathcal{L}^2$ is a finite measure then for any $F \in C_b(\mathbb{R}^2)$,

$$\begin{aligned}
\int_{\mathbb{R}^2} F \circ \mathbf{t_v}\; e \circ \pi_1 \chi_{U_0}\, d\mathcal{L}^2 &= \int_{\mathbb{R}^2} F \circ \mathbf{t_v}\; e \circ \pi_1 \circ \mathbf{t_{-v}} \circ \mathbf{t_v}\; \chi_{U_0} \circ \mathbf{t_{-v}} \circ \mathbf{t_v}\; d\mathcal{L}^2 \\
&= \int_{\mathbb{R}^2} F\; e \circ \pi_1 \circ \mathbf{t_{-v}} \chi_{U_0} \circ \mathbf{t_{-v}}\; d\mathcal{L}^2 \\
&= \int_{\mathbb{R}^2} F\; e \circ \pi_1 \chi_{\{U_0+\mathbf{v}\}}\; d\mathcal{L}^2
\end{aligned} \tag{3.2.7}$$

In the second equality in (3.2.7), we have used the fact the the Lebesgue measure is translation invariant. In the third equality in (3.2.7), we have used $\pi_1 \circ t_{-\mathbf{v}} = \pi_1$. In particular, when $F \equiv 1$ and $U_0 = D_h \setminus \mathcal{E}(h,S,\sigma)$ we get

$$\int_{\{(D_h \setminus \mathcal{E}(h,S,\sigma)) + \mathbf{v}\}} e(s)dp = \int_{D_h \setminus \mathcal{E}(h,S,\sigma)} e(s)dp = 0 \tag{3.2.8}$$

The last equality is due to the fact that $\mathcal{E}(h,S,\sigma)$ *has a full measure with respect to* $e(s)\chi_{D_h}(s,z)\mathcal{L}^2$. $\square$

3.2.1 $\bar{I}[\sigma](h)$ decreases when h is replaced by its non decreasing rearrangement in a class of piecewise constant functions.

Lemma 3.2.2 *Assume* $\sigma \in \mathcal{P}([0,R_0]^2)$ *is absolutely continuous with respect to Lebesgue. Let* h *satisfy (3.2.2) and* $\mathbf{v} = (0,v)$ *such that* $v \neq 0$. *Let* S *be as provided in (3.2.3). Assume there exist* $\bar{h} \in \mathcal{H}_{dom}$, *sets* $V_0 \subset D_{\bar{h}}$ *of non zero* $\mathcal{L}^2$ *measure and* $U_0 \subset \mathcal{E}(h,S,\sigma)$ *such that* $(D_h \setminus U_0)\,\Delta\,(D_{\bar{h}} \setminus V_0)$ *of zero* $\mathcal{L}^2$ *measure,* $V_0 = U_0 + \mathbf{v}$. *Then* $\bar{I}[\sigma](\bar{h}) < \bar{I}[\sigma](h)$.

Proof: Set

$$A = \mathcal{C}(h,S,\sigma) \cap S^{-1}(U_0).$$

By the construction of $\mathcal{C}(h,S,\sigma)$ in remark 3.2.1, $A \subset spt(\sigma)$. As S is injective in $\mathcal{M}_0$, thanks to (3.2.6),

$$S(A) = \mathcal{E}(h,S,\sigma) \cap U_0 = U_0.$$

As $\mathcal{C}(h,S,\sigma)$ has full measure and U_0 of non zero $\mathcal{L}^2$ measure, we obtain

$$\sigma(A) = \sigma\left(\mathcal{C}(h,S,\sigma) \cap S^{-1}(U_0)\right) = \sigma(S^{-1}(U_0)) = S_{\#}\sigma(U_0) = \int_{U_0} e(s)dp > 0$$

Define

$$\bar{S} = S + \mathbf{v}\chi_A.$$

Then

$$\bar{S}(A) = S(A) + \mathbf{v} = U_0 + \mathbf{v} = V_0.$$

Note that as $\pi_1(\mathbf{v}) = 0$,

$$\pi_1 \circ \bar{S} = \pi_1 \circ S + \pi_1 \circ \mathbf{v}\chi_A = \pi_1 \circ S.$$

Let $F \in C_b(\mathbb{R}^2)$. Then

$$
\begin{aligned}
\int_{D_h\setminus U_0} F(s,z)e(s)dsdz &= \int_{(D_h\setminus U_0)\setminus(D_{\bar h}\setminus V_0)} F(s,z)e(s)dsdz \\
&\quad + \int_{(D_h\setminus U_0)\cap(D_{\bar h}\setminus V_0)} F(s,z)e(s)dsdz \\
&= \int_{(D_{\bar h}\setminus V_0)\setminus(D_h\setminus U_0)} F(s,z)e(s)dsdz \\
&\quad + \int_{(D_h\setminus U_0)\cap(D_{\bar h}\setminus V_0)} F(s,z)e(s)dsdz \\
&= \int_{D_{\bar h}\setminus V_0} F(s,z)e(s)dsdz
\end{aligned} \tag{3.2.9}
$$

The second equality in (3.2.9) is due to the fact that $(D_h \setminus U_0)\,\Delta\,(D_{\bar h} \setminus V_0)$ of zero $\mathcal{L}^2$ measure. We observe that $\bar S_{|A} = \mathbf{t_v} \circ S$ and $\bar S_{|A^c} = S$ and so

$$
\begin{aligned}
\int_{\mathbb{R}^2} F \circ \bar S\; \sigma(dq) &= \int_{\mathbb{R}^2\setminus A} F \circ \bar S \sigma(dq) + \int_A F \circ \bar S\; \sigma(dq) \\
&= \int_{\mathbb{R}^2\setminus A} F \circ S\; \sigma(dq) + \int_A F \circ \mathbf{t_v} \circ S\; \sigma(dq)
\end{aligned} \tag{3.2.10}
$$

As $S_\# \sigma = e(s)\chi_{D_h}(s,z)\mathcal{L}^2$, $\mathcal{C}(h,S,\sigma)$ and $\mathcal{E}(h,S,\sigma)$ are of full measure and S bijectively sends $\mathcal{C}(h,S,\sigma)$ onto $\mathcal{E}(h,S,\sigma)$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2\setminus A} F \circ S\; \sigma(dq) &= \int_{\mathcal{C}(h,S,\sigma)\setminus A} F \circ S\; \sigma(dq) \\
&= \int_{\mathcal{E}(h,S,\sigma)\setminus S(A)} F(s,z)e(s)dsdz \\
&= \int_{D_h\setminus S(A)} F(s,z)e(s)dsdz
\end{aligned} \tag{3.2.11}
$$

and

$$
\int_A F \circ \mathbf{t_v} \circ S\; \sigma(dq) = \int_{S(A)} F \circ \mathbf{t_v}\; e(s)\chi_{D_h}(s,z)dsdz \tag{3.2.12}
$$

We combine (3.2.10), (3.2.11), (3.2.12) and $S(A) = U_0 \subset D_h$ to obtain that

$$
\begin{aligned}
\int_{\mathbb{R}^2} F \circ \bar S\; \sigma(dq) &= \int_{D_h\setminus S(A)} F(s,z)e(s)dsdz + \int_{S(A)} F \circ \mathbf{t_v}\; e(s)\chi_{D_h}(s,z)dsdz \\
&= \int_{D_h\setminus U_0} F(s,z)e(s)dsdz + \int_{U_0} F \circ \mathbf{t_v}\; e(s)dsdz
\end{aligned}
$$

This, combined with (3.2.9), (3.2.7) and the fact that $V_0 = U_0 + \mathbf{v}$ yields

$$\begin{aligned}\int_{\mathbb{R}^2} F \circ \bar{S}\ \sigma(dq) &= \int_{D_{\bar{h}} \setminus V_0} F(s,z)e(s)dsdz\ + \int_{U_0+\mathbf{v}} F(s,z)e(s)dsdz \\ &= \int_{D_{\bar{h}} \setminus V_0} F(s,z)e(s)dsdz + \int_{V_0} F(s,z)e(s)dsdz \\ &= \int_{D_{\bar{h}}} F(s,z)e(s)dsdz\end{aligned}$$

As $F \in C_b(\mathbb{R}^2)$ is arbitrary, we conclude that $\bar{S}$ pushes σ forwards to $e(s)\chi_{D_{\bar{h}}}$. Note that

$$\begin{aligned}c(\bar{S}(q), q) &= -\langle \bar{S}(q), q\rangle + \frac{\Omega^2}{2} e^{\frac{1}{2}} \circ \pi_1 \circ \bar{S}(q) \\ &= - < S(q), q > - \chi_A < \mathbf{v}, q > + \frac{\Omega^2}{2} e^{\frac{1}{2}} \circ \pi_1 \circ \bar{S}(q)\end{aligned} \tag{3.2.13}$$

We exploit $\pi_1 \circ \bar{S} = \pi_1 \circ S$ to obtain

$$c(\bar{S}(q), q) = c(S(q), q) - < \mathbf{v}, q > \chi_A$$

Thus, in view of (3.2.5)

$$\bar{I}[\sigma](\bar{h}) - \bar{I}[\sigma](\bar{h}) \leq \int_{\mathbb{R}^2} c(\bar{S}(q), q)\sigma(dq) - \int_{\mathbb{R}^2} c(S(q), q)\sigma(dq) = -\int_A < \mathbf{v}, q > \sigma(dq) \tag{3.2.14}$$

Since $\sigma(A) > 0$, $\sigma << \mathcal{L}^2$ and $A \subset spt(\sigma) \subset [0, R_0]^2$, there exists $A_0 \subset A$ such that $\sigma(A_0) > 0$ and $A_0 \subset (0, R_0)^2$. We have $\langle \mathbf{v}, q\rangle = vq_2 > 0$ for all $q = (q_1, q_2) \in A_0$. In light of (3.2.14), we conclude that $\bar{I}[\sigma](\bar{h}) < \bar{I}[\sigma](h)$. □

Let $n \in \mathbb{N}^*$. For each $1 \leq i \leq n$, we define $\tau_i^n = \frac{i}{n} H$ and consider the following class of functions

$$\mathcal{D}_n = \left\{ h : [0, H] \longrightarrow [0, \infty) \mid h_{|[\tau_{i-1}^n, \tau_i^n)} = h_i,\ \ 1 \leq i \leq n \right\} \tag{3.2.15}$$

Corollary 3.2.3 *Assume $\sigma \in \mathcal{P}([0, R_0]^2)$ is absolutely continuous with respect to Lebesgue. If $h \in \mathcal{D}_n \cap \mathcal{H}_{dom}$ then $\bar{I}[\sigma](h^{\#}) \leq \bar{I}[\sigma](h)$.*

Proof: Let $h \in \mathcal{D}_n$ such that $h_{|[\tau_{i-1},\tau_i)} = h_i$ $1 \leq i \leq n$.

1. Assume that there exist $1 \leq i_0 < j_0 \leq n$ such that $h_{j_0} < h_{i_0}$ and define $\bar{h}(z) = h(z)$ everywhere on $[0, H]$ except $[\tau_{i_0-1}, \tau_{i_0}) \cup [\tau_{j_0-1}, \tau_{j_0})$. $\bar{h}_{|}[\tau_{i_0-1}, \tau_{i_0}) = h_{j_0}$, $\bar{h}_{|}[\tau_{j_0-1}, \tau_{j_0}) = h_{i_0}$. Note that $\bar{h} \in D_n$.

We also observe that

$$
\begin{aligned}
||e^{1/2} \circ \bar{h}||_{L[0,H]} &= \int_0^H e \circ \bar{h}(z)dz \\
&= \sum_{\substack{0<i<n \\ i\neq i_0, i\neq j_0}} \int_{\tau_{i-1}}^{\tau_i} e(h_i)dz + \int_{\tau_{i_0-1}}^{\tau_{i_0}} e(h_{j_0})dz + \int_{\tau_{j_0-1}}^{\tau_{j_0}} e(h_{i_0})dz \\
&= \sum_{\substack{0<i<n \\ i\neq i_0, i\neq j_0}} \int_{\tau_{i-1}}^{\tau_i} e(h_i)dz + \int_{\tau_{i_0-1}}^{\tau_{i_0}} e(h_{i_0})dz + \int_{\tau_{j_0-1}}^{\tau_{j_0}} e(h_{j_0})dz \\
&= \int_0^H e \circ h(z)dz = ||e^{1/2} \circ h||_{L[0,H]}
\end{aligned}
\tag{3.2.16}
$$

We conclude that if $h \in \mathcal{H}_{dom}$ then $\bar{h} \in \mathcal{H}_{dom}$. Set

$$U = (h_{j_0}, h_{i_0}] \times [\tau_{i_0-1}, \tau_{i_0}), \qquad V = (h_{j_0}, h_{i_0}] \times [\tau_{j_0-1}, \tau_{j_0}) \qquad \text{and} \qquad \mathbf{v} = (0, \tau_{j_0} - \tau_{i_0})$$

Note that

$$U \subset D_h, \qquad V \subset D_{\bar{h}}, \qquad V = U + \mathbf{v}.$$

Let $(s, z) \in D_h \setminus U$. If $z \notin [\tau_{i_0-1}, \tau_{i_0}) \cup [\tau_{j_0-1}, \tau_{j_0})$ then $0 \leq s \leq h(z) = \bar{h}(z)$ and $(s, z) \notin V$. We obtain that $(s, z) \in D_{\bar{h}} \setminus V$. If $z \in [\tau_{i_0-1}, \tau_{i_0})$ then on the one hand $(s, z) \notin V$ and on the other hand $0 \leq s \leq h_{j_0} = \bar{h}(z)$ otherwise $(s, z) \in U$. So, again $(s, z) \in D_{\bar{h}} \setminus V$. If $z \in [\tau_{j_0-1}, \tau_{j_0})$ then as $(s, z) \in D_h$, $0 \leq s \leq h_{j_0} < h_{i_0} = \bar{h}(z)$. Again, we obtain that $(s, z) \in D_{\bar{h}} \setminus V$. we conclude that $D_h \setminus U \subset D_{\bar{h}} \setminus V$. A similar reasoning yields $D_{\bar{h}} \setminus V \subset D_h \setminus U$ so that

$$D_h \setminus U = D_{\bar{h}} \setminus V \tag{3.2.17}$$

We assume in the sequel that $h \in \mathcal{H}_{dom} \cap \mathcal{D}_n$.

Let $\mathcal{E}(h, S, \sigma)$ be as in lemma 3.2.2 and define

$$U_0 = U \cap \mathcal{E}(h, S, \sigma) \qquad V_0 = V \cap (\mathcal{E}(h, S, \sigma) + \mathbf{v})$$

so that $V_0 = U_0 + \mathbf{v}$. As $U \in D_h$, we have $D_h = (D_h \setminus U) \cup U$ so that

$$D_h \setminus U_0 = (D_h \setminus U) \cup (U \setminus \mathcal{E}(h, S, \sigma)) \qquad \text{(Disjoint Union).} \tag{3.2.18}$$

Similarly, we use $D_{\bar{h}} = (D_{\bar{h}} \setminus V) \cup V$ to obtain

$$D_{\bar{h}} \setminus V_0 = (D_{\bar{h}} \setminus V) \cup (V \setminus (\mathcal{E}(h, S, \sigma) + \mathbf{v}))$$

By using $V = U + \mathbf{v}$, this becomes

$$D_{\bar{h}} \setminus V_0 = (D_{\bar{h}} \setminus V) \cup ((U \setminus \mathcal{E}(h, S, \sigma)) + \mathbf{v}) \qquad \text{(Disjoint Union).} \tag{3.2.19}$$

We combine (3.2.18) (3.2.19) and (3.2.17) to obtain

$$(D_h \setminus U_0) \,\Delta\, (D_{\bar{h}} \setminus V_0) = (U \setminus \mathcal{E}(h, S, \sigma)) \cup ((U \setminus \mathcal{E}(h, S, \sigma)) + \mathbf{v})$$

And so, as $U \subset D_h$,

$$(D_h \setminus U_0) \,\Delta\, (D_{\bar{h}} \setminus V_0) \subset (D_h \setminus \mathcal{E}(h, S, \sigma)) \cup ((D_h \setminus \mathcal{E}(h, S, \sigma)) + \mathbf{v})$$

In view of (3.2.8) $(D_h \setminus U_0) \,\Delta\, (D_{\bar{h}} \setminus V_0)$ is of zero $\mathcal{L}^2$ measure. By lemma 3.2.2, we conclude that $\bar{I}[\sigma](\bar{h}) < \bar{I}[\sigma](h)$.

2. Let $\Pi(1...n)$ be the set of all the permutations of $1...n$. Let $\bar{\mathfrak{p}} \in \Pi(1...n)$ such that $h_{\bar{\mathfrak{p}}(1)} \leq h_{\bar{\mathfrak{p}}(2)} \leq ... \leq h_{\bar{\mathfrak{p}}(n-1)} \leq h_{\bar{\mathfrak{p}}(n)}$ and let denote by $h^{\bar{\mathfrak{p}}}$ the function defined by $h^{\bar{\mathfrak{p}}}_{|\left(\frac{i-1}{n}, \frac{i}{n}\right)} = h_{\bar{\mathfrak{p}}(i)}$. Note that $h^{\bar{\mathfrak{p}}}$ is a non-decreasing function on $[0, H)$. For any function $F \in L^1([0, H])$, note that

$$\int_{\tau_i}^{\tau_{i+1}} F(h_{\bar{\mathfrak{p}}(i)}) dz = \frac{H}{n} F(h_{\bar{\mathfrak{p}}(i)})$$

and so,

$$\int_0^H F \circ h^{\bar{\mathfrak{p}}} dz = \sum_{i=0}^{n-1} \int_{\tau_i}^{\tau_{i+1}} F(h_{\bar{\mathfrak{p}}(i)}) dz = \sum_{i=0}^{n-1} \int_{\tau_i}^{\tau_{i+1}} F(h_i) dz = \int_0^H F \circ h dz.$$

We conclude that $h^{\#} = h^{\bar{\mathfrak{p}}}$ on $[0, H]$.

3. If h is not monotone non-decreasing then there exist $1 \leq i_0 < j_0 \leq n$ such that $h_{j_0} < h_{i_0}$ and part 1 shows that h is not a minimizer for

$$\inf \left\{ \bar{I}[\sigma](\tilde{h}) : \tilde{h} \in \mathcal{H}_{dom} \cap \mathcal{D}_n, \;\; \tilde{h}_{|\left(\tau_{i-1}^n, \tau_i^n\right)} = h_{\mathfrak{p}(i)}, \;\; \mathfrak{p} \in \Pi(1...n) \right\}$$

As $\Pi(1...n)$ has a finite number of elements, we conclude that the minimizer exists and is reached at $h^{\bar{\mathfrak{p}}}$. $\square$

3.2.2 $\bar{I}[\sigma](h)$ decreases when h is replaced by its non decreasing rearrangement: general case.

In this section $\mathcal{D}_n$ is defined as in (3.2.15) and we set

$$\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n$$

We recall that

$$\mathcal{H}_{dom} = \left\{ h : [0, H] \mapsto [0, \frac{1}{2r_0^2}) : \; ||e^{1/2} \circ h||_{L^1[0,H]} = 2 + Hr_0^2 \right\} \tag{3.2.20}$$

Set

$$\bar{\mathcal{H}}_{dom} := \left\{ g : [0, H] \mapsto [r_0, \infty) : ||g||^2_{L^2[0,H]} = 2 + Hr_0^2 \right\}$$

For simplicity in the presentation, we set $\bar{e}(s) = e^{1/4}(s)$ and note that $\bar{e} : [0 : \frac{1}{2r_0^2}) \longrightarrow [r_0, \infty)$ is continuous and strictly increasing and so bijective. $\bar{e}$ induces a bijective map $\Phi_{\bar{e}} : \mathcal{H}_{\mathrm{dom}} \longrightarrow \bar{\mathcal{H}}_{\mathrm{dom}}$ defined by $g \longmapsto \bar{e} \circ g$ with inverse $\Phi_{\bar{e}^{-1}}$.

Definition 3.2.4 *A function $h : [0, H] \longrightarrow \mathbb{R}$ is said to be a rearrangement of a function g if for every measurable function F such that $F \circ g \in L^1[0, H]$, we have $F \circ h \in L^1[0, H]$ and*

$$\int_0^H F \circ g \, dz = \int_0^H F \circ h \, dz.$$

In other words,

$$g_{\#}\chi_{[0,H]} = h_{\#}\chi_{[0,H]}.$$

We denote by $h^{\#}$ the rearrangement of h that is monotone and non-decreasing as provided by lemma A.0.14 (i).

Lemma 3.2.5 *(i) $\Phi_{\bar{e}^{-1}}(\bar{h}^{\#}) = \Phi_{\bar{e}^{-1}}(\bar{h})^{\#}$ for all $\bar{h} \in \bar{\mathcal{H}}_{dom}$.*
(ii) Let $g \in \bar{\mathcal{H}}_{dom} \cap L^4[0,H]$. Then there exists $g_n \in \mathcal{D} \cap \bar{\mathcal{H}}_{dom}$ and a constant $c > 0$ such that $\{g_n\}_{n=1}^{\infty}$ converges to g in $L^2[0,H]$ and $||g_n||_{L^4[0,H]} < c$.
(iii) the monotone rearrangement operator $\#$ is continuous from $\bar{\mathcal{H}}_{dom}$ to $\bar{\mathcal{H}}_{dom}$ endowed with the $L^2[0,H]-$ norm.
(iv) If g_n and $g \in \bar{\mathcal{H}}_{dom}$ such that $\{g_n\}_{n=1}^{\infty}$ converges to g in $L^2[0,H]$ then (up to a subsequence) $\Phi_{\bar{e}^{-1}}(g_n)$ converges to $\Phi_{\bar{e}^{-1}}(g)$ almost everyhere.

Proof: (i) is a direct consequence of lemma A.0.14 (iii).
1. We proceed to show part (ii) of the lemma.
(a) Assume $0 \leq f \in L^4[0,H]$. Define $g_n = f\chi_{\{f<n\}}$. Then $g_n \in L^{\infty}[0,H]$ $0 \leq g_n \leq f$

$$g_n \longrightarrow f \ a.e \qquad \text{and} \qquad |g_n - f|^4 \leq 16|f|^4 \in L^1[0,H]$$

By the dominated convergence theorem, we obtain that g_n converges to f in $L^4[0,H]$.

(b) Assume $f \in L^{\infty}[0,H]$. we extend f to be zero outside of $[0,H]$ and call the new function $\tilde{f}$. For each $\varepsilon > 0$, set $g_\varepsilon = \tilde{f} * \eta_\varepsilon$ where η_ε are the standard mollifiers . Then $g_\varepsilon \in C[0,H]$, g_ε converges a.e to f and $||g_\varepsilon||_{L^2[0,H]} \leq ||f||_{L^2[0,H]}$. Moreover,

$$|g_\varepsilon(x)| \leq \int_{\mathbb{R}} |\tilde{f}(y)\eta_\varepsilon(x-y)|dy \leq ||\tilde{f}||_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \eta_\varepsilon(x-y)dy = ||\tilde{f}||_{L^\infty(\mathbb{R})} =: M$$

Note that

$$|g_\varepsilon - f|^4 \leq 8\left(|g_\varepsilon|^4 + |f|^4\right) \leq 8M^2 + 8|f|^4 \in L^1[0,H]$$

By the dominated convergence theorem, g_ε converges to f in $L^4[0,H]$

(c) Assume $h \in C[0,H]$ and set

$$\delta(n) = \max_{1 \le i \le n} \max_{[\tau_{i-1}^n, \tau_i^n]} |h(x) - h_i|.$$

where h_i is the minimum value of h on $[\tau_{i-1}^n, \tau_i^n]$. Since h is uniformly continuous $\lim_{n\to\infty} \delta(n) = 0$.

Set $\tilde{h}_n = \sum_{i=1}^n h_i \chi_{[\tau_{i-1}^n, \tau_i^n)}$. Then $\tilde{h}_n \in \mathcal{D}$. By construction, $0 \le \tilde{h}_n \le h$. We easily check that

$$||h - \tilde{h}_n||^4_{L^4[0,H]} = \sum_{i=1}^n \int_{\tau_i^n}^{\tau_{i+1}^n} |h - h_i|^4 \le H\delta^4(n)$$

and so,

$$\lim_{n\to\infty} ||h - \tilde{h}_n||_{L^4[0,H]} = 0.$$

Now let $g \in \bar{\mathcal{H}}_{dom} \cap L^4[0,H]$. Without loss of generality we assume $g \ge 0$. From the results established in (a), (b), (c), there exists a sequence of functions $\{g_n\}_{n=1}^\infty \subset \mathcal{D}$ such that $\{g_n\}_{n=1}^\infty$ converges to g in $L^4[0,H]$(and thus in $L^2[0,H]$) and $||g_n||_{L^2[0,H]} \le ||g||_{L^2[0,H]}$. We lose no generality if we assume that $g_n \in \mathcal{D} \cap \bar{\mathcal{H}}_{dom}$. This follows from the following fact :if $\{a_n\}_{n=1}^\infty \subset L^2[0,H]$ is such that $||a_n||_{L^2[0,H]} \le ||a||_{L^2[0,H]}$ and $\{a_n\}_{n=1}^\infty$ converges to a in $L^2[0,H]$ then $b_n = \frac{||a||_{L^2[0,H]}}{||a_n||_{L^2[0,H]}} a_n$ converges to a in $L^2[0,H]$ and $||b_n||_{L^2[0,H]} = ||a||_{L^2[0,H]}$. As $\{g_n\}_{n=1}^\infty$ converges to g in $L^4[0,H]$, we obtain that $\{g_n\}_{n=1}^\infty$ is bounded in $L^4[0,H]$. This concludes the proof of (ii).

2. The continuity of the Monotone rearrangement operator is immediate from lemma A.0.14 (ii).

3. Assume $\{g_n\}_{n=1}^\infty$ converge to g in $L^2[0,H]$. There exists a subsequence $\{n_k\}_{k=0}^\infty$ of integer such that $\{g_{n_k}\}_{k=1}^\infty$ converge to g a.e. Since $\bar{e}^{-1}$ is continuous, it follows that $\Phi_{\bar{e}^{-1}}(g_{n_k})$ converges to $\Phi_{\bar{e}^{-1}}(g)$ a.e. $\square$

Proposition 3.2.6 *Let $\sigma \in \mathcal{P}([0,R_0]^2)$ absolutely continuous with respect to Lebesgue measure. Assume $h \in \mathcal{H}_{dom}$ such that $||e \circ h||_{L^1[0,H]} < \infty$. Then*

$$h^\# \in \mathcal{H}_{dom} \qquad \textit{and} \qquad \bar{I}[\sigma](h^\#) \le \bar{I}[\sigma](h)$$

Proof: By lemma A.0.14 (iii), $(\sqrt{e}\circ h)^{\#} = \sqrt{e}\circ h^{\#}$. The fact that $h^{\#}\in\mathcal{H}_{dom}$ is a consequence of the following:

$$||\sqrt{e}\circ h||_{L^1[0,H]} = ||(\sqrt{e}\circ h)^{\#}||_{L^1[0,H]} = ||\sqrt{e}\circ h^{\#}||_{L^1[0,H]}$$

The first equality in the equation above is a consequence of the definition of the rearrangement of functions. Let $\bar{h}\in\bar{\mathcal{H}}_{dom}$ such that $h = \Phi_{\bar{e}^{-1}}(\bar{h})$. As

$$||\bar{h}||^4_{L^4[0,H]} = ||\Phi_{\bar{e}}(h)||^4_{L^4[0,H]} = ||\sqrt[4]{e}\circ h||^4_{L^4[0,H]} = ||e\circ h||_{L^1[0,H]} < \infty.$$

By lemma 3.2.5 (ii), there exists $\{\bar{h}_n\}_{n=1}^{\infty}\in\mathcal{D}\cap\bar{\mathcal{H}}_{dom}$ such that $\{\bar{h}_n\}_{n=1}^{\infty}$ converges to $\bar{h}$ in $L^2[0,H]$ and a constant c such that $||\bar{h}_n||^4_{L^4[0,H]} < c$. Set $h_n := \Phi_{\bar{e}^{-1}}(\bar{h}_n)$. By Lemma 3.2.5 (iv) we may assume without loss of generality that h_n converges to h a.e. Note that

$$||e\circ h_n||_{L^1[0,H]} = ||\sqrt[4]{e}\circ h_n||^4_{L^4[0,H]} = ||\bar{e}\circ\Phi_{\bar{e}^{-1}}(\bar{h}_n)||^4_{L^4[0,H]} = ||\bar{h}_n||^4_{L^4[0,H]} < c \quad (3.2.21)$$

And so, by using lemma 3.1.2 first and then corollary 3.1.4, we have that $\bar{I}[\sigma](h_n)$ converges to $\bar{I}[\sigma](h)$. By lemma 3.2.5 (i), we obtain

$$h_n^{\#} = \Phi_{\bar{e}^{-1}}(\bar{h}_n^{\#})$$

This, combined with the fact that $||\bar{h}_n||^4_{L^4[0,H]} < c$, gives

$$||e\circ h_n^{\#}||_{L^1[0,H]} = ||\sqrt[4]{e}\circ h_n^{\#}||^4_{L^4[0,H]} = ||\Phi_{\bar{e}}(h_n^{\#})||^4_{L^4[0,H]} = ||\bar{h}_n||^4_{L^4[0,H]} < c \quad (3.2.22)$$

As $\bar{h}_n$ converges to $\bar{h}$ in $L^2[0,H]$, the continuity of the operator # ensures that $\{\bar{h}_n^{\#}\}_{n=1}^{\infty}$ converges to $\bar{h}^{\#}$ in $L^2[0,H]$. Lemma 3.2.5 (iv) again ensures without loss of generality that $\{h_n^{\#}\}_{n=1}^{\infty}$ converges to $h^{\#}$ a.e. In view of (3.2.22), lemma 3.1.2 and corollary 3.1.4 guarantees that $\bar{I}[\sigma](h_n^{\#})$ converges to $\bar{I}[\sigma](h^{\#})$.

By Corollary 3.2.3, we obtain $I[\sigma](h_n^{\#}) \leq I[\sigma](h_n)$. Hence, in the limit $I[\sigma](h^{\#}) \leq I[\sigma](h)$ $\square$

Proposition 3.2.7 *Assume $\sigma \in \mathcal{P}\left([0,R_0]^2\right)$ is absolutely continuous with respect to Lebegue. Then $\bar{I}[\sigma]$ has a minimizer in $\mathcal{H}_{dom}$.*

Proof: let $\{h_n\}_{n=1}^{\infty}$ be a minimizing sequence in

$$\inf_{h \in \mathcal{H}_{dom}} \bar{I}[\sigma](h)$$

We assume without loss of generality that $\bar{I}[\sigma](h_n) \leq \bar{I}[\sigma](h_0) < \infty$. By lemma 3.1.2, this implies that $||e \circ h_n||_{L^1[0,H]} < c$ for some constant c. In light of Proposition 3.2.6 , we can assume without loss of generality $\{h_n\}_{n=1}^{\infty}$ is a sequence of monotone non decreasing functions. By Helly's theorem, we may assume that $\{h_n\}_{n=1}^{\infty}$ converges some h. By using lemma 3.1.2 and then lemma 3.1.5, we obtain that $h \in \mathcal{H}_{dom}$. Corollary 3.1.4 ensures that h is a minimizer for $I[\sigma]$. $\square$

3.3 Duality Methods and Monge-Ampere Problem.

In this section, we show the existence and uniqueness for the minimization problem (1.4.4) by coming up with a dual problem. This provides a unique solution to the Monge Ampere equation (3.3.1). Futhermore, this dual formulation helps establish a better regularity result for the domain D_h.

Given σ a probability measure in $[0,R_0]^2$, we consider a system of PDEs, where the unknown are functions

$$\Psi : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}, \qquad P : [0, \frac{1}{2r_0^2}) \times [0,H] \longrightarrow \mathbb{R}, \qquad h : [0,H] \longrightarrow [0, \frac{1}{2r_0^2}).$$

We impose that Ψ and P are Legendre transforms of each other and these functions solve the system of equations

$$\begin{cases} e(\frac{\partial \Psi}{\partial \Upsilon}) \det(\nabla^2_{\Upsilon,Z}\Psi) = \sigma \\ \nabla\Psi(spt(\sigma)) = D_h \\ P(h(z),z) = \frac{\Omega^2 r_0^2}{2(1-2r_0^2 h(z))} \quad \text{on} \quad \{h>0\} \end{cases} \tag{3.3.1}$$

Definition 3.3.1 *We assume that $\sigma = \rho\mathcal{L}^2$. Let $P : [0, \frac{1}{2r_0^2}) \times [0,H] \longrightarrow \mathbb{R}$ and $\Psi : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ be Legendre transforms of each other and $h : [0,H] \longrightarrow [0, \frac{1}{2r_0^2})$. We say that P, Ψ and h solve equation (3.3.1) in a weak sense if*

$$\begin{cases} \nabla_{\Upsilon,Z}\Psi_{\#}\sigma = e(s)\chi_{D_h}(s,z)\mathcal{L}^2 \\ P(h(z),z) = \frac{\Omega^2 r_0^2}{2(1-2r_0^2 h(z))} \quad on \quad \{h>0\} \end{cases} \tag{3.3.2}$$

For $\sigma \in \mathcal{P}([0,R_0]^2)$, we say that P, Ψ and h solve equation (3.3.1) in the dual weak sense if

$$\begin{cases} \nabla P_{\#}\left(e(s)\chi_{D_h}(s,z)\mathcal{L}^2\right) = \sigma \\ P(h(z),z) = \frac{\Omega^2 r_0^2}{2(1-2r_0^2 h(z))} \quad on \quad \{h>0\} \end{cases} \tag{3.3.3}$$

Our main result is the following

Theorem 3.3.2 *Let $R_0 > 0$. Let σ be a probability measure on $\mathbb{R}^2$ such that the support of σ is contained in $[0,R_0]^2$. Then (3.3.1) admits a unique solution $(\bar{\Psi}, \bar{P}, \bar{h})$. $(\bar{\Psi}, \bar{P})$ is obtained as the maximizer in (3.3.11) and $\bar{h}$ is monotone and obtained as the minimiser in (3.3.5). Moreover, if the support of σ is contained in $[\frac{1}{R_0}, R_0] \times [0, R_0]$ then $\partial D_{\bar{h}}$ is Lipschitz continuous.*

3.3.1 Primal and Dual formulation of the problem.

Let σ be a probability measure such that $\mathrm{spt}(\sigma) \subset \Delta \subset [0,R]^2$. Thanks to (3.2.1), we rewrite

$$\bar{I}[\sigma](h) = \inf_{\gamma \in \Gamma(\sigma,\, e(s)\chi_{D_h}(s,z))} I(h,\gamma)$$

Here,

$$I(h,\gamma) := \int_{D_h\times\Delta} C(p,q)\gamma(dp,dq) \quad \text{with} \quad C(p,q) = -\langle p,q\rangle + \frac{\Upsilon}{2r_0^2} - \Omega\sqrt{\Upsilon} + \frac{r_0^2\Omega^2}{2(1-2r_0^2 s)}, \tag{3.3.4}$$

and so,

$$\inf_{h\in\mathcal{H}_{dom}} \bar{I}[\sigma](h) = \inf_{(h,\gamma)\in\mathfrak{L}_\sigma} I(h,\gamma) \tag{3.3.5}$$

where

$$\mathfrak{L}_\sigma = \Big\{(h,\gamma) : h\in\mathcal{H}_0, \int_{D_h} e(s)dsdz = 1, \gamma\in\Gamma(e(s)\chi_{D_h}\mathcal{L}^2,\sigma)\Big\} \tag{3.3.6}$$

$\mathcal{H}_0$ is the set of all borel measurable functions $h:[0,H]\longrightarrow[0,\frac{1}{2r_0^2})$. To study the minimization problem in (3.3.5), we will introduce what will turn out to be its dual formulation by setting:

$$J[\sigma](\Psi,P) = \int_{\mathbb{R}^2}\Big(\frac{\Upsilon}{2r_0^2} - \Omega\sqrt{\Upsilon} - \Psi\Big)\sigma(dq) + j(P); \quad j(P) = \inf_{h\in\mathcal{H}_0}\int_0^H \Pi_P(h(z),z)dz. \tag{3.3.7}$$

$J[\sigma]$ is defined on

$$\mathcal{U} := \Big\{(\Psi,P)\in C(\mathbb{R}^2_+)\times C(\bar{\Delta}_{r_0}) : P(p)+\Psi(q)\geq\langle p,q\rangle \text{ for all } (p,q)\in\Delta_{r_0}\times\mathbb{R}^2_+\Big\} \tag{3.3.8}$$

To $P:\Delta_{r_0}\longrightarrow\mathbb{R}$ we have associated

$$\Pi_P(\rho,z) = \int_0^\rho\Big(\frac{\Omega^2 r_0^2}{2(1-2sr_0^2)} - P(s,z)\Big)e(s)ds \quad \text{for} \quad 0\leq 2r_0^2\rho<1. \tag{3.3.9}$$

We observe that if $P_1\leq P_2$ then $\Pi_{P_1}\geq\Pi_{P_2}$ and also that if P is a constant function that is equal to C in (3.3.9) then

$$\Pi_C(\rho,z) := \Pi_P(\rho,z) = \frac{\Omega^2 r_0^6(1-\rho r_0^2)\rho}{2(1-2\rho r_0^2)^2} - \frac{Cr_0^4\rho}{1-2\rho r_0^2} \tag{3.3.10}$$

The dual problem we will be looking at is the following:

$$\sup_{(\Psi,P)\in\mathcal{U}_0} J[\sigma](\Psi,P) \tag{3.3.11}$$

3.3.2 Existence of a minimizer for $\Pi_P(\cdot, z)$ and Twist condition.

Let's denote by $\mathcal{U}_0$ the subset of $\mathcal{U}$ consisting of pairs (Ψ, P) such that

$$P(p) = \sup_{q \in \bar{\Delta}} \left(\langle p, q \rangle - \Psi(q) \right), \qquad p \in \Delta_{r_0} \quad \text{and} \quad \Psi(q) = \sup_{p \in \Delta_{r_0}} \left(\langle p, q \rangle - P(p) \right) \qquad q \in \mathbb{R}^2_+. \tag{3.3.12}$$

We note that if P and Ψ satisfy (3.3.12) then P and Ψ are convex lower semicontinuous as supremum of convex and lower semicontinuous functions and

$$\partial_\cdot P(p) \subset \bar{\Delta} \text{ for all } p \in \Delta_{r_0} \qquad \text{and} \qquad \partial_\cdot \Psi(q) \subset \bar{\Delta}_{r_0} \text{ for all } q \in \mathbb{R}^2_+ \tag{3.3.13}$$

As a consequence P and Ψ are lipschitz continuous with

$$Lip P \leq R_0 \qquad \text{and} \qquad Lip \Psi \leq \max\{H, \frac{1}{2r_0^2}\} \tag{3.3.14}$$

By abuse of notation we denote the expression at the right handside of the first equation in (3.3.12) by Ψ^* and the one at the right handside of second equation in (3.3.12) by P^*.

We consider functions $P : \Delta_{r_0} \to \mathbb{R}$ lipschitz such that

$$0 \leq \frac{\partial P}{\partial z}(s, z) \leq R_0 \qquad \text{and} \qquad 0 \leq \frac{\partial P}{\partial s}(s, z) \leq R_0 \tag{3.3.15}$$

Lemma 3.3.3 *Let $A \in \mathbb{R}_+$. Suppose $P_* : \Delta_{r_0} \to \mathbb{R}$ such that $P_* \leq P_*(0,0) + A$. Then there exists a constant M_{P_*} depending on $P_*(0,0)$ such that $2r_0^2 M_{P_*} < 1$ and*

$$\sup_{0 \leq z \leq H} \sup_{0 \leq 2r^2 \rho < 1} \{\rho \mid \Pi_{P_*}(\rho, z) \leq 0\} \leq M_{P_*}. \tag{3.3.16}$$

Furthermore, M_{P_*} is monotone nondecreasing in $P_*(0,0)$.

Proof: Since $P_* \leq P_*(0,0) + A$,

$$\Pi_{P_*} \geq \Pi_{P_*(0,0)+A} \tag{3.3.17}$$

Let $(\rho, z) \in \Delta_{r_0}$ such that $\Pi_{P_*}(\rho, z) \leq 0$. Then, by (3.3.17) and (3.3.10)

$$0 \geq \Pi_{P_*}(\rho, z) \geq \frac{\Omega^2 r_0^6 (1 - \rho r_0^2)\rho}{2(1 - 2\rho r_0^2)^2} - \frac{(P_*(0,0) + A) r_0^4 \rho}{1 - 2\rho r_0^2}$$

That is,

$$\frac{\Omega^2 r_0^6 (1 - \rho r_0^2)\rho}{2(1 - 2\rho r_0^2)} \leq \Big[P_*(0,0) + A\Big] r_0^4 \rho \tag{3.3.18}$$

Therefore, for any $z \in [0, H]$ fixed,

$$\{\rho \mid \Pi_{P_*}(\rho, z) \leq 0\} \subset \{\rho \mid \frac{\Omega^2 r_0^6 (1 - \rho r_0^2)\rho}{2(1 - 2\rho r_0^2)} \leq \Big[P_*(0,0) + A\Big] r_0^4 \rho\}$$

Set

$$M_{P_*} := \sup_{0 \leq 2r_0^2 \rho < 1} \left\{\rho \mid \frac{\Omega^2 r_0^6 (1 - \rho r_0^2)\rho}{2(1 - 2\rho r_0^2)} \leq \Big[P_*(0,0) + A\Big] r_0^4 \rho\right\}$$

We note that M_{P_*} is independent of z and so (3.3.16) is satisfied. Moreover, $\rho = 0$ satisfies (3.3.18) so that $M_{P_*} \in \mathbb{R}$. As ρ tends $1/(2r_0^2)^+$ the expression at the left hand side of (3.3.18) tends to $+\infty$ and the one at the right hand side of (3.3.18) goes to a finite value. We then conclude that $2r_0^2 M_{P_*} < 1$. The expression of M_{P_*} above shows that it is monotone nondecreasing in $P_*(0,0)$. □

Remark 3.3.4 *Let P be a continuous function on $\bar{\Delta}_{r_0}$ and K be a compact subset of $\bar{\Delta}_{r_0}$. Then, as the integrand in Π_P is continuous on K, Π_P is continuous on K. Let c_0 be an upper bound for $(s, z) \longmapsto e(s)$ on K. Then we have the estimate*

$$|\Pi_{P_n}(\rho, z) - \Pi(\rho, z)| \leq c_0 \int_0^{\frac{1}{2r_0^2}} |P_n(s, z) - P(s, z)|\, ds \leq \frac{c}{2r_0^2} ||P_n - P||_{L^\infty(K)}$$

And so, if we assume that $\{P_n\}_{n=1}^{\infty}$ is a sequence of continuous functions on $\bar{\Delta}_{r_0}$ such that $\{P_n\}_{n=1}^{\infty}$ converges uniformly to P on K then we easily show that $\{\Pi_{P_n}\}_{n=1}^{\infty}$ converges uniformly to Π_P on K.

Lemma 3.3.5 *Assume $P_n, P : \bar{\Delta}_{r_0} \to \mathbb{R}$ satisfy the hypotheses in lemma 3.3.3 and are continuous .*

(i) Given $z \in [0, H]$, the set $Argmin\Pi_P(\cdot, z)$ consisting of h minimizing $\Pi_P(\cdot, z)$ over $[0, 1/(2r_0^2))$ is non empty. Moreover,

$$\bigcup_{0 \le z \le H} Argmin\Pi_P(\cdot, z) \subset [0, M_P] \tag{3.3.19}$$

where M_P is as in lemma 3.3.3.

(ii) Suppose $\{P_n\}_{n=1}^{\infty}$ converges uniformly to P on $\bar{\Delta}_{r_0}$. Then

$$2r_0^2 \sup_n M_{P_n} < 1. \tag{3.3.20}$$

If in addition, $\{z_n\}_{n=1}^{\infty} \subset [0, H]$ converges to z and we assume that $h_n \in Argmin\Pi_{P_n}(\cdot, z_n)$ and that $\{h_n\}_{n=1}^{\infty}$ converges to h then

$$\lim_{n \to \infty} \Pi_{P_n}(h_n, z_n) = \Pi_P(h, z) \qquad \text{and} \qquad h \in Argmin\Pi_P(\cdot, z). \tag{3.3.21}$$

In particular, for each $z \in [0, H]$ the set $Argmin\Pi_P(\cdot, z)$ is compact subset of $\mathbb{R}$.

(iii) Assume in addition that $P(\rho, \cdot)$ is Lipschitz and the first equation in (3.3.15) holds a.e on $(0, H)$ for each ρ fixed. Let $z_1, z_2 \in [0, H]$ be such that $z_1 < z_2$. If $h_i \in Argmin\Pi_P(\cdot, z_i)$ $i = 1, 2$ then $h_1 \le h_2$.

Proof: (i) Let $z \in [0, H]$. As $\Pi_P(0, z) = 0$, in light of lemma 3.3.3, minimizing $\Pi_P(\cdot, z)$ over $[0, 1/(2r_0^2))$ is equivalent to minimizing $\Pi_P(\cdot, z)$ over $[0, M_P]$. As observed in remark 3.3.4, $\Pi_P(\cdot, z)$ is continuous on $[0, M_P]$. Hence, it admits a minimum there and $Argmin\Pi_P(\cdot, z) \subset [0, M_P]$. This establishes (3.3.19).

(ii) The convergence propriety of $\{P_n\}_{n=1}^{\infty}$ ensures that $\{P_n(0, 0)\}_{n=1}^{\infty}$ converges to $P(0, 0)$ and so $\{P_n(0, 0)\}_{n=1}^{\infty}$ is bounded above by one of its terms say $P_{n_0}(0, 0)$ or $P(0, 0)$. The monotonicity result in lemma 3.3.3 ensures that $M_{P_n} \le M_{P_{n_0}} < \frac{1}{2r_0^2}$ or $M_{P_n} \le M_P < \frac{1}{2r_0^2}$ for all $n \ge 1$. Thus, (3.3.20) holds.

Let $\{z_n\}_{n=1}^{\infty} \subset [0, H]$ be a sequence converging to z and assume $h_n \in Argmin\Pi_{P_n}(\cdot, z_n)$ and is such that $\{h_n\}_{n=1}^{\infty}$ converges to h and let $\rho \in [0, 1/(2r_0^2))$. We choose M such that $M_P, \sup_n M_{P_n}, \rho \leq M < \frac{1}{2r_0^2}$ so that $K := [0, M] \times [0, H]$ is compact subset of Δ_{r_0}. We use the fact that h_n minimizes $\Pi_{P_n}(\cdot, z_n)$, that $\{\Pi_{P_n}\}_{n=1}^{\infty}$ converges uniformly to Π_P on K and that Π_P is continuous (cfr. remark 3.3.4) to get

$$\Pi_P(h, z) = \lim_{n\to\infty} \Pi_{P_n}(h_n, z_n) \leq \lim_{n\to\infty} \Pi_{P_n}(\rho, z_n) = \Pi_P(\rho, z).$$

Since this holds for any $\rho \in [0, 1/(2r_0^2))$, we have that $h \in Argmin\Pi_P(\cdot, z)$. In particular, the previous conclusion holds for $z_n = z$ and $P_n = P$ for all n, to yield that $Argmin\Pi_P(\cdot, z)$ is a closed subset of $[0, M_P]$ which implies that it is compact.

(iii) For each $z \in [0, H]$, $\Pi_P(\cdot, z)$ is differentiable on $(0, 1/(2r_0^2))$ and its derivative is the integrand of Π_P. As $P(\rho, \cdot)$ is Lipschitz , $\partial\Pi_P/\partial\rho(\rho, \cdot)$ is differentiable almost everywhere on $(0, H)$ and

$$\frac{\partial^2\Pi_P}{\partial z\partial\rho}(\rho, z) = -e(\rho)\frac{\partial P}{\partial z}(\rho, z) \leq 0. \tag{3.3.22}$$

We have used the first equation in (3.3.15). This means that Π_P satisfies the so–called twist condition. Let $z_i \in [0, H]$ and $h_i \in Argmin\Pi_P(\cdot, z_i)$ $i = 1, 2$. We use the minimality condition on h_1, h_2 and the fact that $P(\rho, \cdot)$ is lipschitz to obtain

$$0 \leq \Big(\Pi_P(h_2, z_1) - \Pi_P(h_1, z_1)\Big) + \Big(\Pi_P(h_1, z_2) - \Pi_P(h_2, z_2)\Big) = -\int_{h_1}^{h_2} d\rho \int_{z_1}^{z_2} \frac{\partial^2\Pi_P}{\partial z\partial\rho}(\rho, z)dz. \tag{3.3.23}$$

If $z_1 < z_2$, then we use (3.3.22) and (3.3.23) to get $h_1 \leq h_2$. $\square$

Remark 3.3.6 *Let $z \in [0, H]$ and $h \in Argmin\Pi_P(\cdot, z)$. If $h > 0$ then $\partial\Pi/\partial\rho(h, z) = 0$ that is,*

$$P(h, z) = \frac{\Omega^2 r_0^2}{2(1 - 2hr_0^2)}. \tag{3.3.24}$$

Let $P : \Delta_{r_0} \to \mathbb{R}$ be such that $Argmin\Pi_P(\cdot, z)$ is compact for each $z \in [0, H]$. We define

$$h^+(z) = \max_{Argmin\Pi_P(\cdot,z)} h, \qquad h^-(z) = \min_{Argmin\Pi_P(\cdot,z)} h \tag{3.3.25}$$

Lemma 3.3.7 *Assume P satisfies the hypotheses in lemma 3.3.5 and the first equation in (3.3.15). Then, the following hold:*

(i) h^- is lower semi-continuous and h^+ is upper semi-continuous .

(ii) h^-, h^+ is monotone nondecreasing.

(iii) Let $z_1, z_2 \in [0, H]$ be such that $z_1 < z_2$. Then $h^+(z_1) \leq h^-(z_2)$.

(iv) h^- is left continuous and h^+ is right continuous.

(v) Let $z \in [0, H)$. If h^- is continuous at z then $h^-(z) = h^+(z)$.

Proof: (i) Let $\{z_n\}_{n=1}^{\infty} \subset [0, H]$ be a sequence converging to z and set $a = \liminf_{n\to\infty} h^-(z_n)$. We assume without loss of generality that $a = \lim_{n\to\infty} h^-(z_n)$ by passing to a subsequence if necessary. We use corollary 3.3.5 (ii) to get that $a \in Argmin\Pi_P(\cdot, z)$ The definition of $h^-(z)$ ensures that $a \geq h^-(z)$. So, h^- is lower semi-continuous. We obtain the upper semi-continuiuty of h^+ in a similar way.
The monotonicity of h^- and h^+ comes from corollary 3.3.5 (iii).
(iii) is immediate again from corollary 3.3.5 (iii).
(iv) We use the fact that h^- is monotone nondecreasing and lower semi-continuous to obtain that h^- is left continuous. A similar argument gives that h^+ is right continuous.
(v) Let $z_0 \in [0, H)$ such that $h^-(z_0) < h^+(z_0)$. We note that, as h^- is monotone nondecreasing, it has a right limit. For $\delta > 0$ small enough, we use Part (iii) to obtain that $h^+(z_0) \leq h^-(z_0 + \delta)$ and so

$$h^-(z_0) < h^+(z_0) \leq \lim_{\delta\to 0^+} h^-(z_0 + \delta).$$

This implies that h^- is discontinuous at z_0 which proved (v). $\square$

Corollary 3.3.8 *There exists a countable set $\mathcal{N} \subset [0, H]$ such that for every $z \notin \mathcal{N}$, $Argmin \Pi_P(\cdot, z)$ has a unique element.*

Proof: Since $h^- : [0, H] \to \mathbb{R}$ is monotone nondecreasing, h^- has a countable set of points of discontinuity . We use lemma 3.3.7(v) to conclude that there exists a countable set $\mathcal{N} \subset [0, H]$ such that for $z \notin \mathcal{N}$, $h^-(z) = h^+(z)$. $\square$

Remark 3.3.9 *If P is Lipschitz and satisfies (3.3.15). Then*

$$P(p) - P(0,0) \leq |P(p) - P(0,0)| \leq R_0|p| \leq R_0 \left(\frac{1}{2r_0^2} + H \right) \tag{3.3.26}$$

for any $p \in \Delta_{r_0}$. And so, P satisfies the hypotheses in lemma 3.3.3, lemma 3.3.5 and lemma 3.3.7.

Consequently,

$$\Pi_P(h^-(z), z) \leq \Pi_P(h(z), z).$$

for all $h \in \mathcal{H}_0$. Thus,

$$\int_0^H \Pi_P(h^-(z), z) dz \leq \int_0^H \Pi_P(h(z), z) dz.$$

As h is arbitrary in $\mathcal{H}_0$, we conclude that h^- is a minimizer in the second equation of (3.3.7) .

$\square$

Remark 3.3.10 *Let P be Lipschitz and satisfying (3.3.15) and $\bar{h}(z)$ a minimizer in the second equation of (3.3.7). By (3.3.19),*

$$0 \leq \bar{h}(z) \leq M_P < \frac{1}{2r_0^2}. \tag{3.3.27}$$

for all $z \in [0, H]$. Moreover, if $\{P_n\}_{n=1}^{\infty}$ is a sequence of Lipschitz functions uniformly convergent on Δ_{r_0} and satisfying (3.3.15) with minimizers $\left\{\bar{h}_n\right\}_{n=1}^{\infty}$ in the second equation of (3.3.7) when P is replaced by P_n then by (3.3.27) and (3.3.20)

$$0 \leq \bar{h}_n(z) \leq \sup_n M_{P_n} < \frac{1}{2r_0^2} \tag{3.3.28}$$

for all $z \in [0, H]$ and all $n \geq 1$. $\square$

3.3.3 Existence of a minimizer for the functional I.

Remark 3.3.11 *Let $(\Psi, P) \in \mathcal{U}_0$. As Δ and Δ_{r_0} are bounded, by (3.3.13), P and Ψ are Lipschitz cfr(3.3.14) and P satisfies (3.3.15). If in addition $P(0,0) = 0$ then, in view of (3.3.26)*

$$|P(p)| \leq R_0\left(\frac{1}{2r_0^2} + H\right) =: R_0 H_0. \tag{3.3.29}$$

We note that $0 \leq \langle p, q\rangle \leq R_0 H_0$ for $q \in \bar{\Delta}$ and $p \in \Delta_{r_0}$. This combined with the second equation in (3.3.12) and (3.3.29) yields that Ψ is bounded on $\bar{\Delta}$ more precisely

$$-2R_0H_0 < -R_0H_0 \leq \Psi(q) \leq 2H_0R_0$$

for $q \in \bar{\Delta}$. As a consequence

$$\Big|\int_\Delta \Big(\frac{\Upsilon}{2r_0^2} - \Omega\sqrt{\Upsilon} - \Psi\Big)\sigma(dq)\Big| \leq \int_\Delta \Big(\frac{\Upsilon}{2r_0^2} + \Omega\sqrt{\Upsilon}\Big)\sigma(dq) + 2H_0R_0 =: C(R_0) + 2H_0R_0 < \infty.$$

Lemma 3.3.12 *Let $C_0 \in \mathbb{R}$ and $\sigma \in \mathcal{P}([0, R_0]^2)$. There exists a constant C_1 depending only on C_0 (and the data R_0, H_0) satisfying the following: whenever $(P, \Psi) \in \mathcal{U}_0$ with $P(0,0) = 0$, $\lambda \in \mathbb{R}$ such that $-C_0 \leq J[\sigma](\Psi + \lambda, P - \lambda)$ then $|\lambda| \leq C_1$.*

Proof: By (3.3.29) $-L_0H_0 < P(p)$ for $p \in \Delta_{r_0}$ so that

$$\Pi_{P-\lambda} \leq \Pi_{-H_0L_0-\lambda}$$

Therefore, if $J[\sigma](\Psi + \lambda, P - \lambda) \geq -C_0$ then

$$-C_0 \leq -\lambda + \int_\Delta \Big(\frac{\Upsilon}{2r_0^2} - \Omega\sqrt{\Upsilon} - \Psi \Big) \sigma(dq) + \int_0^H \Pi_{-H_0 L_0 - \lambda}(h(z), z) dz$$

for all $h \in \mathcal{H}_0$. Hence, using $C(R_0)$ as given in remark 3.3.11 and setting h to be a constant function $\bar{h}_0$ we obtain

$$-C_0 \leq -\lambda + C(R_0) + 2H_0 R_0 + \int_0^H \Pi_{-H_0 L_0 - \lambda}(\bar{h}_0, z) dz$$

We use (3.3.10) to get

$$-\bar{C}_0 := -C_0 - C(R_0) - 2H_0 R_0 \leq -\lambda + \frac{\Omega^2 r_0^6 (1 - \bar{h}_0 r_0^2) \bar{h}_0 H}{2(1 - 2\bar{h}_0 r_0^2)^2} - \frac{(-H_0 L_0 - \lambda) H r_0^4 \bar{h}_0}{1 - 2\bar{h}_0 r_0^2}$$

we rewrite this as

$$\lambda \Big(1 - H \frac{r_0^4 \bar{h}_0}{1 - 2\bar{h}_0 r_0^2} \Big) \leq \bar{C}_0 + H \frac{r_0^4 H_0 L_0 \bar{h}_0}{1 - 2\bar{h}_0 r_0^2} + H \frac{\Omega^2 r_0^6 (1 - \bar{h}_0 r_0^2) \bar{h}_0}{2(1 - 2\bar{h}_0 r_0^2)^2}. \tag{3.3.30}$$

set $\bar{h}_0 = 0$ in (3.3.30) then

$$\lambda \leq \bar{C}_0 \tag{3.3.31}$$

when the constant value of $\bar{h}_0$ is chosen in $[0, \frac{1}{2r_0^2})$ (for instance close enough to $\frac{1}{2r_0^2}$) so that the factor of λ in (3.3.30) is negative then there exists a constant $\bar{C}_1$ such that

$$\lambda \geq \bar{C}_1 \tag{3.3.32}$$

We combine (3.3.31) and (3.3.32) to get the result. □

We recall that for each $P : \Delta_{r_0} \longrightarrow \mathbb{R}$ lipschitz,

$$j(P) = \inf_{h \in \mathcal{H}_0} \int_0^H \Pi_P(h(z), z) dz. \tag{3.3.33}$$

and

$$h^-(z) = \min_{Argmin \Pi_P(\cdot, z)} h$$

Lemma 3.3.13 *(i) Let P be a lipschitz function satisfying (3.3.15) on Δ_{r_0}. Then, h^- is the unique minimizer in (3.3.33) (up to a set of zero lebesgue measure).*

(ii) Assume that $\{P_n\}_{n=1}^{\infty}$ is a sequence of Lipschitz functions on Δ_{r_0} satisfying (3.3.15) such that $\{P_n\}_{n=1}^{\infty}$ converges uniformly to P. Then

$$j(P_n) \quad \textit{converges to} \quad j(P).$$

Proof: i) The function h^- is a minimizer in (3.3.33) as stated in remark 3.3.9 . Let $\hat{h}$ be another minimizer. Then

$$\int_0^H \Pi_P(h^-(z), z)dz = \int_0^H \Pi_P(\hat{h}(z), z)dz. \tag{3.3.34}$$

Since $h^-(z) \in Argmin\Pi_P(\cdot, z)$,

$$\Pi_P(h^-(z), z) \leq \Pi_P(\hat{h}(z), z) \tag{3.3.35}$$

We use (3.3.34) and (3.3.35) to obtain that

$$\Pi_P(h^-(z), z) = \Pi_P(\hat{h}(z), z) \quad a.e \tag{3.3.36}$$

Therefore $\hat{h}(z) \in Argmin\Pi_P(\cdot, z) \quad a.e.$ By corollary 3.3.8 we obtain that $\hat{h}(z) = h^-(z) \quad a.e.$

ii) We first note that as $\{P_n\}_{n=1}^{\infty}$ is uniformly lipschitz and converges uniformly to P, we have that P is lipschitz.

Let h_n^- be defined as in (3.3.25) when P is replaced by P_n.

By Helly's theorem there exists a subsequence of $\{h_n^-\}_{n=1}^{\infty}$ that we denote again by $\{h_n^-\}_{n=1}^{\infty}$ and h monotone nondecreasing such that $\{h_n^-\}_{n=1}^{\infty}$ converges to h pointwise.

In view of (3.3.28), there exists a positive constant M such that

$$2r_0 M < 1 \qquad \text{and} \qquad 0 \leq h_n^-(z) \leq M.$$

for all $z \in [0, H]$ and all $n \geq 1$. $\{\Pi_{P_n}\}_{n=1}^{\infty}$ converges uniformly to Π_P on $[0, M] \times [0, H]$ by remark 3.3.4 . As $\{h_n^-\}_{n=1}^{\infty}$ converges pointwise to h, by (3.3.21),

$\{\Pi_{P_n}(h_n^-(z), z)\}_{n=1}^{\infty}$ converges pointwise to $\Pi_P(h(z), z)$ and $h(z) \in Argmin\Pi_P(\cdot, z)$ for all $z \in [0, H]$. And so, as P is Lipschitz, $Argmin\Pi_P(\cdot, z)$ reduces to $h^-(z)$ a.e so that

$$h^- = h \quad a.e.$$

As $\{P_n\}_{n=1}^{\infty}$ converges uniformly to P, $\{P_n\}_{n=1}^{\infty}$ is uniformly bounded by a constant C and we have

$$|\Pi_{P_n}| \leq \Pi_{-|P_n|} \leq \Pi_{-C}$$

Therefore, since $0 \leq h_n(z) \leq M$ and $\Pi_{-C}(., z)$ is monotone non decreasing,

$$|\Pi_{P_n}(h_n(z), z)| \leq \Pi_{-C}(h_n(z), z) \leq \Pi_{-C}(M, z) =: const \tag{3.3.37}$$

By the Lebesgue dominated convergence theorem, we use the fact that $\{\Pi_{P_n}(h_n(z), z)\}_{n=1}^{\infty}$ converges pointwise to $\Pi_P(h^-(z), z)$ and (3.3.37) to obtain

$$j(P_n) \quad \text{converges to} \quad j(P).$$

□

Remark 3.3.14 *We recall that if P has the representation in the first equation of (3.3.12) for some $\Psi \in C(\bar{\Delta})$ then subgradient $\partial_{.}P(p) \subset \bar{\Delta} \subset [0, R_0] \times [0, R_0]$ and so P is R_0-Lipschitz.*

Lemma 3.3.15 *Let $\sigma \in \mathcal{P}([0, R_0]^2)$ and $(P_0, \Psi_0) \in \mathcal{U}_0$.*
(i) Let $g \in C_c(\mathbb{R}^2)$. For any $\delta \in (-1, 1)$, we set

$$\Psi_\delta = \Psi_0 + \delta g, \qquad \textit{and} \qquad P_\delta = \Psi_\delta^*.$$

Then, $\{P_\delta\}_{-1<\delta<1} \subset C(\bar{\Delta}_{r_0})$ *and*

$$||P_\delta - P_0||_\infty \leq |\delta|||g||_\infty \qquad \textit{and} \qquad \lim_{\delta\to 0}\frac{P_\delta(p) - P_0(p)}{\delta} = -g(\nabla P_0(p)) \qquad (3.3.38)$$

for all $p \in \mathrm{dom}(\nabla P_0)$.

(ii) Likewise, let $f \in C_c(\mathbb{R}^2)$. *For any* $\bar{\delta} \in (-1,1)$, *we set*

$$P_{\bar{\delta}} = P_0 + \bar{\delta} f \qquad \textit{and} \qquad \Psi_{\bar{\delta}} = P^*_{\bar{\delta}}.$$

Then $\Psi_{\bar{\delta}} \subset C([0,R]^2)$ *and*

$$||\Psi_{\bar{\delta}} - \Psi_0||_\infty \leq |\delta|||f||_\infty \qquad \textit{and} \qquad \lim_{\delta\to 0}\frac{\Psi_{\bar{\delta}}(q) - \Psi_0(q)}{\delta} = -f(\nabla \Psi_0(q)) \qquad (3.3.39)$$

for all $q \in \mathrm{dom}(\nabla \Psi_0)$.

Proof: Let $p \in \Delta_{r_0}$ and $\delta > 0$.

$$P_\delta(p) = \sup_{q\in\bar{\Delta}} \left(\langle p, q\rangle - \Psi_0(q) - \delta g(q)\right).$$

As $\bar{\Delta}$ is compact, there exists $q_\delta \in \bar{\Delta}$ such that

$$P_\delta(p) = \langle p, q_\delta\rangle - \Psi_\delta(q_\delta) - \delta g(q_\delta) \leq P_0(p) - \delta g(q_\delta) \qquad (3.3.40)$$

When $\delta = 0$, we use the equality in (3.3.40) to obtain

$$P_0(p) = \langle p, q_0\rangle - \Psi_0(q_0) - \delta g(q_0) + \delta g(q_0) \leq P_\delta(p) + \delta g(q_0) \qquad (3.3.41)$$

We combine (3.3.40) and (3.3.41) to obtain

$$\delta g(q_\delta) \leq P_0(p) - P_\delta(p) \leq \delta g(q_0) \qquad (3.3.42)$$

so that

$$||P_\delta - P_0||_\infty \leq |\delta|||g||_\infty$$

We rewrite (3.3.42) as

$$g(q_\delta) \leq \frac{P_0(p) - P_\delta(p)}{\delta} \leq g(q_0) \qquad (3.3.43)$$

And so, to prove the second equation in (3.3.38), as g is continuous, it suffices to show that

$$\lim_{\delta \to 0} q_\delta = q_0 = \nabla P_0(p) \qquad (3.3.44)$$

if P_0 is differentiable at p. Note that the first equation in (3.3.40) shows that $q_0 \in \partial P_0(p)$ and so if P_0 is differentiable at p then $q_0 = \nabla P_0(p)$. let $\{\delta_n\}_{n=1}^\infty \subset (-1,1)$ be a sequence nonzero numbers that converges to 0. Then, as $q_{\delta_n} \in \bar{\Delta}$ there exists a subsequence of $\{\delta_n\}_{n=1}^\infty$ still denoted by $\{\delta_n\}_{n=1}^\infty$ such that $\{q_{\delta_n}\}_{n=1}^\infty$ converges to $\bar{q} \in \bar{\Delta}$. We substitute q_δ by q_{δ_n} in the first equation of (3.3.40) and let δ_n go to 0. As, $\{P_{\delta_n}\}_{n=1}^\infty$ and $\{\Psi_{\delta_n}\}_{n=1}^\infty$ are uniformly convergent we obtain in the limit

$$P_0(p) = \langle p, \bar{q} \rangle - \Psi_0(\bar{q})$$

If p is a point of differentiability of P_0 then $\bar{q} = q_0 = \nabla P_0(p)$ and so the whole sequence $\{q_{\delta_n}\}_{n=1}^\infty$ converges to q_0 and we conclude (3.3.44).

A similar reasonning leads to (3.3.39). $\square$

Lemma 3.3.16 *Let $\{P_\delta\}_{-1<\delta<1} \in C(\bar{\Delta}_{r_0})$ uniformly lipschitz continuous and satisfying the first equation in (3.3.15) such that $\{P_\delta\}_{-1<\delta<1}$ satisfies the first equation in (3.3.38). For each $-1 < \delta < 1$ we define h_δ^- as in (3.3.25). Then $\{h_\delta^-\}_{\substack{-1<\delta<1 \\ \delta \neq 0}}$ converges to h_0^-. Moreover, if $\{\delta_n\} \subset (-1,1)$ converges to 0 then*

$$\limsup_{n\to\infty} \left| \frac{1}{\delta_n} \left[j(P_{\delta_n}) - j(P_0) - \int_0^H \int_0^{h_0^-(z)} (P_{\delta_n}(s,z) - P_0(s,z)) e(s) ds dz \right] \right| = 0$$

Proof: As (3.3.38) holds, $\{P_\delta\}_{\substack{-1<\delta<1 \\ \delta \neq 0}}$ converges uniformly to P_0 as δ goes to 0.

Fix $z \in [0,H]$ and let $\{\delta_n\}_{n=1}^\infty \subset (-1,1)$ be sequence of non zero numbers converging to 0. As $\{P_{\delta_n}\}_{n=1}^\infty$ uniformly converges to P_0, we have that $\{h_{\delta_n}^-(z)\}_1^\infty$ is bounded in light of (3.3.28) and so, without loss of generality we assume that $\{h_{\delta_n}^-(z)\}_{n=1}^\infty$ converges. If z is a continuity point for h_0^- then lemma 3.3.7 (v) ensures that $h_0^-(z)$ is the unique element of $Argmin \Pi_{P_0}(\cdot, z)$ and so by using lemma 3.3.5 (ii) we obtain

$$\lim_{n\to\infty} h^-_{\delta_n}(z) = h^-_0(z).$$

As $\{\delta_n\}_{n=1}^{\infty}$ is arbitrary, we obtain

$$\lim_{\delta\to 0} h^-_{\delta}(z) = h^-_0(z). \tag{3.3.45}$$

In light of corollary 3.3.8, (3.3.45) holds for almost every $z \in [0, H]$.

Fix $\delta \in (-1, 1)$. By definition of $h^-_0(z)$ we have $\Pi_{P_0}(h^-_0(z), z) \leq \Pi_{P_0}(h^-_\delta(z), z)$ and so

$$\begin{aligned} \Pi_{P_0}(h^-_0(z), z) - \Pi_{P_\delta}(h^-_\delta(z), z) &\leq \Pi_{P_0}(h^-_\delta(z), z) - \Pi_{P_\delta}(h^-_\delta(z), z) \\ &= \int_0^{h^-_\delta(z)} (P_\delta(s, z) - P_0(s, z))e(s)ds. \end{aligned} \tag{3.3.46}$$

Similarly, we establish that

$$\begin{aligned} \Pi_{P_\delta}(h^-_\delta(z), z) - \Pi_{P_0}(h_0(z), z) &\leq \Pi_{P_\delta}(h^-_0(z), z) - \Pi_{P_0}(h^-_0(z), z) \\ &= -\int_0^{h^-_0(z)} (P_\delta(s, z) - P_0(s, z))e(s)ds. \end{aligned} \tag{3.3.47}$$

Let again $\{\delta_n\}_{n=1}^{\infty} \subset (-1, 1)$ converging to 0. We use the definition of j in (3.3.33), along with (3.3.46), (3.3.47) to obtain that

$$\begin{aligned} \int_0^H \mathrm{d}z \int_0^{h^-_0(z)} (P_{\delta_n}(s, z) - P_0(s, z))e(s)\mathrm{d}s &\leq j(P_0) - j(P_{\delta_n}) \\ &\leq \int_0^H dz \int_0^{h^-_{\delta_n}(z)} (P_{\delta_n}(s, z) - P_0(s, z))e(s)\mathrm{d}s. \end{aligned} \tag{3.3.48}$$

As $\{P_{\delta_n}\}_{n=1}^{\infty}$ converges uniformly to P_0, and satisfies (3.3.15), by (3.3.28), there exists M such that $2r_0^2 M < 1$ and

$$0 \leq h^-_{\delta_n}(z) \leq M \tag{3.3.49}$$

for all $z \in [0, H]$ and $n \geq 1$. This ensures that the integrals in (3.3.48) are finite for $n \geq 1$.

We rewrite (3.3.48) as

$$\begin{aligned} 0 \leq j(P_0) - j(P_{\delta_n}) - &\int_0^H dz \int_0^{h_0^-(z)} (P_{\delta_n}(s,z) - P_0(s,z))e(s)ds \\ &\leq \int_0^H \int_{h_0^-(z)}^{h_{\delta_n}^-(z)} (P_{\delta_n}(s,z) - P_0(s,z))e(s)dsdz \end{aligned} \tag{3.3.50}$$

We use the fact e is bounded on $[0, M]$, the first equation in (3.3.38) and apply the Lebesgue dominated convergence, using (3.3.45) and (3.3.49) to obtain that

$$\begin{aligned} \limsup_{n\to\infty} &\left| \int_0^H dz \int_{h_0^-(z)}^{h_{\delta_n}^-(z)} \frac{(P_{\delta_n}(s,z) - P_0(s,z))e(s)}{\delta_n} ds \right| \\ &\leq \max_{[0,H]} e ||g||_\infty \limsup_{n\to\infty} \int_0^H |h_{\delta_n}^-(z) - h_0^-(z)| dz \leq 0 \end{aligned} \tag{3.3.51}$$

□

Proposition 3.3.17 *Let $\sigma \in \mathcal{P}([0, R_0]^2)$.*

(i) The set of maximizers $\mathcal{M}$ of $J[\sigma]$ over $\mathcal{U}$ is such that $\mathcal{M} \cap \mathcal{U}_0$ is non empty. $\mathcal{U}_0$ is defined by (3.3.12).

(ii) $I(\gamma, h) \geq J[\sigma](\Psi, P)$ for all $(\Psi, P) \in \mathcal{U}_0$ and all $(\gamma, h) \in \mathfrak{L}_\sigma$. The equality holds if and only if ∇P pushes forward $e(s)\chi_{D_h}(s,z)\mathcal{L}^2$ onto σ and $h(z)$ minimizes $\Pi_P(., z)$ for almost every $z \in [0, H]$. If in addition σ is absolutely continuous with respect to Lebesgue $\mathcal{L}^2$, then the equality holds as well if and only if $\nabla\Psi$ pushes σ onto $e(s)\chi_{D_h}(s,z)\mathcal{L}^2$.

(iii) I has a unique minimizer (γ_0, h_0) over $\mathfrak{L}_\sigma$. Moreover, if $(\Psi_0, P_0) \in \mathcal{U}_0$ maximizes $J[\sigma]$ on $\mathcal{U}$ then $J[\sigma](\Psi_0, P_0) = I(\gamma_0, h_0)$, $\mathbf{id} \times \nabla P_0$ pushes $e(s)\chi_{D_{h_0}}(s,z)\mathcal{L}^2$ onto

γ_0 and h_0 is monotone non decreasing on $[0, H]$ satisfying

$$2(1 - 2r_0^2 h_0(z))P(h_0(z), z) = r_0^2\Omega^2 \text{ on } \{h_0 > 0\} \tag{3.3.52}$$

If σ is absolutely continuous with respect to Lebesgue then $\nabla\Psi_0 \times \mathbf{id}$ pushes σ onto γ_0 and

$$\nabla\Psi_0 \circ \nabla P_0 = \mathbf{id} \quad e(s)\chi_{D_{h_0}}(s,z)\mathcal{L}^2 \quad a.e \quad \nabla P_0 \circ \nabla\Psi_0 = \mathbf{id} \quad a.e \quad \sigma \tag{3.3.53}$$

(iii) $J[\sigma]$ has a unique maximizer (Ψ_0, P_0) on $\mathcal{U}_0$ in the sense that if $J[\sigma](\Psi_0, P_0) = J[\sigma](\Psi_1, P_1)$ then $P_1 = P_0 \quad e(s)\chi_{D_{h_0}}(s,z)\mathcal{L}^2 \quad a.e$ and $\Psi_1 = \Psi_0 \quad a.e \quad \sigma$.

Proof: 1. Set

$$\bar{c}_0 = \sup_{(p,q)\in\Delta\times\Delta_{r_0}} \langle p, q\rangle, \qquad \bar{P}_0(p) = \bar{c}_0, \quad \bar{\Psi}_0(q) = 0$$

so that

$$(\bar{\Psi}_0, \bar{P}_0) \in \mathcal{U} \quad \text{ and } \quad -C_0 := J[\sigma](\bar{\Psi}_0, \bar{P}_0) - 1 \text{ is finite.}$$

Let $\{(\bar{\Psi}_n, \bar{P}_n)\}_{n=1}^{\infty} \subset \mathcal{U}$ be a maximizing sequence for $J[\sigma]$ over $\mathcal{U}$.
We note that whenever $(\bar{\Psi}, \bar{P}) \in \mathcal{U}$, by the double convexification trick (cfr. [46] Page 51), we have

$$(\bar{P}^*, \bar{P}**) \in \mathcal{U}_0 \text{ and } J[\sigma](\bar{\Psi}, \bar{P}) \le J[\sigma](\bar{P}^*, \bar{P}**).$$

This shows, on the one hand, that if the set of maximizers $\mathcal{M}$ of $J[\sigma]$ over $\mathcal{U}$ is non empty then so is $\mathcal{M} \cap \mathcal{U}_0$ and, on the other hand, that we may assume without loss of generality that $\{(\bar{P}_n, \bar{\Psi}_n)\}_{n=1}^{\infty}$ is contained in $\mathcal{U}_0$. We assume so and set

$$\Psi_n = \bar{\Psi}_n + \bar{P}_n(0,0), \quad \lambda_n = -\bar{P}_n(0,0), \quad P_n = \bar{P}_n - \bar{P}_n(0,0)$$

we easily check that $\{(\Psi_n, P_n)\}_{n=1}^{\infty} \subset \mathcal{U}_0$ and

$$\lim_{n\to\infty} J[\sigma](\Psi_n + \lambda_n, P_n - \lambda_n) = \lim_{n\to\infty} J[\sigma](\bar{\Psi}_n, \bar{P}_n) = \sup_{\mathcal{U}} J[\sigma]$$

and so, for n large enough

$$-C_0 \leq J[\sigma](\Psi_n + \lambda_n, P_n - \lambda_n). \tag{3.3.54}$$

Therefore, as $P_n(0,0) = 0$ by lemma 3.3.12 we obtain that $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is bounded. Hence, up to a subsequence, $\{\lambda_n\}_{n=1}^{\infty}$ converges to a real number λ_*.
We substitute P_n for P and Ψ_n for Ψ in remark 3.3.11 and note that the sequences $\{P_n\}_{n=1}^{\infty} \subset C(\bar{\Delta}_{r_0})$ and $\{\Psi_n\}_{n=1}^{\infty} \subset C(\bar{\Delta})$ are uniformly bounded and uniformly Lipschitz. We then use Ascoli- Arzerla to conclude that there exists a subsequence of $\{(\Psi_n, P_n)\}_{n=1}^{\infty}$ converging uniformly to some $(\Psi_*, P_*) \in C(\bar{\Delta}) \times C(\bar{\Delta}_{r_0})$.
In the sequel, we assume without loss of generality that

$$\{\lambda_n\}_{n=1}^{\infty} \text{ converges to } \lambda_* \text{ and } \{(\Psi_n, P_n)\}_{n=1}^{\infty} \text{ converges uniformly to } (\Psi_*, P_*).$$

We set

$$P_0 := P_* - \lambda_*, \quad \Psi_0 := \Psi_* + \lambda_*$$

Therefore

$$\{(\bar{\Psi}_n, \bar{P}_n)\}_{n=1}^{\infty} \text{ converges uniformly to } (\Psi_0, P_0).$$

We note that $\{\bar{P}_n\}_{n=1}^{\infty}$ are lipschitz and satifies (3.3.15) (cfr remark 3.3.11).

We use the fact that $\{\bar{\Psi}_n)\}_{n=1}^{\infty}$ converges uniformly to Ψ_0, and lemma 3.3.13 (ii) to obtain that

$$\{J[\sigma](\bar{\Psi}_n, \bar{P}_n)\}_{n=1}^{\infty} \text{ converges to } J[\sigma](\Psi_0, P_0).$$

This established that (Ψ_0, P_0) is a maximizer of $J[\sigma]$ over $\mathcal{U}$.

2. Let $(\Psi, P) \in \mathcal{U}_0$ and $(\gamma, h) \in \mathfrak{L}_\sigma$. Then Ψ, P are Lipschitz by remark 3.3.11 and $P(p) + \Psi(q) \geq \langle p, q \rangle$.
Recall

$$J[\sigma](\Psi,P)=\int_{D_h\times\mathbb{R}^2_+}\Big(\frac{\Upsilon}{2r_0^2}-\Omega\sqrt{\Upsilon}-\Psi\Big)\gamma(dp,dq)+j(P) \tag{3.3.55}$$

We note that

$$\begin{aligned} j(P)\leq\int_0^H\Pi_P(h(z),z)dz&=\int_0^H\int_0^{h(z)}\Big(\frac{\Omega^2r_0^2}{2(1-2sr_0^2)}-P(s,z)\Big)e(s)dsdz\\ &=\int_{D_h}\Big(\frac{\Omega^2r_0^2}{2(1-2sr_0^2)}-P(s,z)\Big)e(s)dsdz, \end{aligned} \tag{3.3.56}$$

and as $\gamma\in\Gamma(e(s)\chi_{D_h}\mathcal{L}^2,\sigma)$,

$$\int_{D_h}\Big(\frac{\Omega^2r_0^2}{2(1-2sr_0^2)}-P(s,z)\Big)e(s)dsdz=\int_{D_h\times\mathbb{R}^2_+}\Big(\frac{\Omega^2r_0^2}{2(1-2sr_0^2)}-P(p)\Big)\gamma(dp,dq) \tag{3.3.57}$$

We combine (3.3.55-3.3.57) with $P(p)+\Psi(q)\geq\langle p,q\rangle$ to get

$$J[\sigma](\Psi,P)\leq\int_{D_h\times\mathbb{R}^2_+}\left(-\langle p,q\rangle+\frac{\Upsilon}{2r_0^2}-\Omega\sqrt{\Upsilon}+\frac{r_0^2\Omega^2}{2(1-2r_0^2\rho)}\right)\gamma(dp,dq)=I(h,\gamma) \tag{3.3.58}$$

Note that equality holds in (3.3.58) if and only if equality holds in (3.3.56) and $P(p)+\Psi(q)=\langle p,q\rangle$ for γ almost every (p,q) . The first condition means that $h(z)\in Argmin\Pi_P(\cdot,z)$ for almost every $z\in[0,H]$ by using lemma 3.3.13 (i). As the first projection (marginal) of γ is absolutely continuous with respect to $\mathcal{L}^2$, the second condition means that $q=\nabla P(p)$ for γ almost every (p,q) and so, γ is concentrated on the graph of ∇P. This implies that γ is the push forward of $e(s)\chi_{D_h}(s,z)\mathcal{L}^2$ by $\mathbf{id}\times\nabla P$. If σ is absolutely continuous with respect to $\mathcal{L}^2$ then the condition $P(p)+\Psi(q)=\langle p,q\rangle$ for γ almost every (p,q) implies that γ is the push-forward of σ by $\nabla\Psi\times\mathbf{id}$.

3. Assume that $(P_0,\Psi_0)\in\mathcal{U}_0$ is a maximizer of $J[\sigma]$ over $\mathcal{U}$.

(a) Let $g\in C_c(\mathbb{R}^2)$ $\{(P_\delta,\Psi_\delta)\}$ as defined in lemma 3.3.15. As P_0 is lipschitz, the second equation in (3.3.38) holds almost everywhere with respect to $\mathcal{L}^2$.

By the Lebesgue dominated convergence theorem, (3.3.38) implies that

$$\lim_{n\to\infty}\int_0^H dz\int_0^{h_0(z)}\frac{(P_{\delta_n}(s,z)-P_0(s,z))e(s)}{\delta_n}ds=-\int_0^H\int_0^{h_0(z)}g(\nabla P_0(s,z))e(s)dsdz \tag{3.3.59}$$

We note that

$$\frac{J[\sigma](P_{\delta_n},\Psi_{\delta_n})-J[\sigma](P_0,\Psi_0)}{\delta_n}=-\int_\Delta gd\sigma+\frac{j(P_{\delta_n})-j(P_0)}{\delta_n} \tag{3.3.60}$$

We use the fact that $\{\delta_n\}_{n=1}^\infty$ is an arbitrary sequence that converges to 0, lemma 3.3.16 and combine (3.3.59) and (3.3.60) to get

$$\lim_{\delta\to 0}\frac{J[\sigma](P_\delta,\Psi_\delta)-J[\sigma](P_0,\Psi_0)}{\delta}=-\int_\Delta gd\sigma+\int_0^H dz\int_0^{h_0(z)}g(\nabla P(s,z))e(s)ds. \tag{3.3.61}$$

Since (P_0,Ψ_0) maximizes $J[\sigma]$ over $\mathcal{U}$ and $(P_\delta,\Psi_\delta)\in\mathcal{U}$, (3.3.61) implies that

$$\int_\Delta gd\sigma=\int_0^H dz\int_0^{h_0(z)}g(\nabla P_0(p))e(s)dp. \tag{3.3.62}$$

(3.3.62) holds for any $g\in C_c(\mathbb{R}^2)$ which means that ∇P_0 pushes $e(s)\chi_{D_{h_0}}(s,z)\mathcal{L}^2$ forward to σ.

(b) Let $f\in C_c(\mathbb{R}^2)$ and $\{(P_{\bar\delta},\Psi_{\bar\delta})\}$ as defined in lemma 3.3.15. As Ψ_0 is lipschitz, in view of (3.3.39) the Lebesgue dominated convergence implies that

$$\lim_{n\to\infty}\int_\Delta\frac{(\Psi_{\bar\delta_n}(q)-\Psi_0(q))}{\bar\delta_n}d\sigma=-\int_\Delta f(\nabla\Psi_0(p))d\sigma \tag{3.3.63}$$

We also have that

$$\int_0^H dz\int_0^{h_0(z)}\frac{(P_{\bar\delta_n}(s,z)-P_0(s,z))e(s)}{\bar\delta_n}ds=\int_0^H\int_0^{h_0(z)}f(s,z)e(s)dsdz \tag{3.3.64}$$

We use lemma 3.3.16 and (3.3.64) to obtain that

$$\lim_{n\to\infty}\frac{j(P_{\bar\delta_n})-j(P_0)}{\bar\delta_n}=\int_0^H\int_0^{h_0(z)}f(s,z)e(s)dsdz \tag{3.3.65}$$

As,

$$\frac{J[\sigma](P_{\bar{\delta}_n}, \Psi_{\bar{\delta}_n}) - J[\sigma](P_0, \Psi_0)}{\bar{\delta}_n} = -\int_\Delta \frac{(\Psi_{\bar{\delta}_n}(q) - \Psi_0(q))}{\bar{\delta}_n} d\sigma + \frac{j(P_{\bar{\delta}_n}) - j(P_0)}{\bar{\delta}_n}$$

We combine (3.3.63) and (3.3.65) to obtain

$$\lim_{\delta \to 0} \frac{J[\sigma](P_{\bar{\delta}}, \Psi_{\bar{\delta}}) - J[\sigma](P_0, \Psi_0)}{\bar{\delta}} = -\int_\Delta f(\nabla \Psi_0(q)) d\sigma + \int_0^H dz \int_0^{h_0(z)} f(s,z) e(s) ds. \tag{3.3.66}$$

(P_0, Ψ_0) is a maximizer (3.3.66) yields that

$$\int_\Delta f(\nabla \Psi_0(q)) d\sigma = \int_0^H dz \int_0^{h_0(z)} f(s,z) e(s) ds. \tag{3.3.67}$$

The arbitrariness of $f \in C_c(\mathbb{R}^2)$ in (3.3.67) implies that $\nabla \Psi_0$ pushes σ forward to $e(s)\chi_{D_{h_0}}(s,z)\mathcal{L}^2$.

4. Let $(P_0, \Psi_0) \in \mathcal{U}_0$ be a maximizer of $J[\sigma]$ over $\mathcal{U}$. Here, we denotes by $h_0(z)$ is the smallest element of $Argmin \Pi_{P_0}(\cdot, z)$ and we set

$$\gamma_0 := (\mathbf{id} \times \nabla P_0)_{\#} e(s) \chi_{D_{h_0}}(s,z) \mathcal{L}^2$$

that is, γ_0 is the pushed forward of $e(s)\chi_{D_{h_0}}\mathcal{L}^2$ by $(\mathbf{id} \times \nabla P_0)$. Then, by part (ii) we have $I(h_0, \gamma_0) = J[\sigma](P_0, \Psi_0)$ which ensures that (h_0, γ_0) is a minimizer in (3.3.5). Let $(\bar{h}, \bar{\gamma})$ be another minimizer in (3.3.5). Then $I(\bar{h}, \bar{\gamma}) = I(h_0, \gamma_0)$ and so $I(\bar{h}, \bar{\gamma}) = J[\sigma](P_0, \Psi_0)$. Again by part (ii), $\bar{\gamma}$ is the pushed forward of $e(s)\chi_{D_{\bar{h}}}\mathcal{L}^2$ by $(\mathbf{id} \times \nabla P_0)$ and $\bar{h}(z) \in Argmin \Pi_{P_0}(\cdot, z)$ for a.e $z \in [0, H]$. We use corollary 3.3.8 to obtained that $\bar{h}(z) = h_0(z)$ a.e. These prove that the minimizer in (3.3.5) is unique. By remark 3.3.6, $2(1 - 2r_0^2 h_0(z)) P_0(h_0(z), z) = 2r_0^2 \Omega^2$ on $\{h_0 > 0\}$.

If in addition σ is absolutely continuous with respect to Lebesgue then again by using part (ii) of this theorem, we have that (h_0, γ_1) is a minimizer in (3.3.5) with $\gamma_1 := (\nabla \Psi_0 \times \mathbf{id})_{\#}\sigma$. The uniquness of the minimizer in I guarantees that $\gamma_1 = \gamma_0$.

5. Let (P_0, Ψ_0) be a maximizer of $J[\sigma]$ on $\mathcal{U}_0$ as in part 4. We have $P_0(p) + \Psi_0(q) = \langle p, q \rangle$ for γ_0 almost every (p, q) (see part 2). Assume σ is absolutely continuous with

respect to Lebesgue. Then

$$P_0(p) + \Psi_0(\nabla P_0(p)) = \langle p, \nabla P_0(p)\rangle \qquad e(s)\chi_{D_h}(s,z)\mathcal{L}^2 \ a.e \tag{3.3.68}$$

$$P_0(\nabla\Psi_0(q)) + \Psi_0(q) = \langle \nabla\Psi_0(q), q\rangle \qquad \sigma \ a.e \tag{3.3.69}$$

Let N be the set of points where Ψ is not differentiable. Then $\sigma(N) = 0$. As $\nabla P_{0\#}e(s)\chi_{h_0}(s,z)\mathcal{L}^2 = \sigma$, we have that the preimage of N by ∇P_0 is of zero measure with respect to $e(s)\chi_{D_{h_0}}(s,z)\mathcal{L}^2$. In view of (3.3.68) and (3.3.69), we conclude that $\nabla\Psi_0 \circ \nabla P_0(p) = p \quad e(s)\chi_{D_{h_0}}\mathcal{L}^2 - a.e.$ A similar argument shows the second equation of (3.3.53).

6. Assume (P_1, Ψ_1) is another maximizer of $J[\sigma]$ in $\mathcal{U}_0$. Then

$$\gamma_0 = (\mathbf{id} \times \nabla P_0)_\# e(s)\chi_{D_{h_0}}(s,z)\mathcal{L}^2 = (\mathbf{id} \times \nabla P_1)_\# e(s)\chi_{D_{h_0}}(s,z)\mathcal{L}^2.$$

This implies that $\nabla P_0 = \nabla P_1 \ e(s)\chi_{D_{h_0}}(s,z)\mathcal{L}^2$-a.e and so $\mathcal{L}^2$-a.e on D_{h_0}. As, D_{h_0} is connected and P_0 and P_1 are lipschitz continuous satisfying (3.3.52), we conclude that $P_1 = P_0$ on D_{h_0} and without loss of generality $P_1 = P_0$ on Δ_{r_0}. Consequently, $\Psi_1 = \Psi_2$ on Δ.

3.3.4 Regularity property of the domain D_h.

In this section we consider the functions P lipschitz that satisfy

$$0 \le \frac{\partial P}{\partial s}(s,z) \le R_0 \qquad \text{and} \qquad \frac{1}{R_0} \le \frac{\partial P}{\partial z}(s,z) \le R_0 \tag{3.3.70}$$

As a consequence, $Argmin\Pi_P(\cdot, z)$ is compact. We recall that

$$h^+(z) = \max_{Argmin\Pi_P(\cdot,z)} h, \qquad h^-(z) = \min_{Argmin\Pi_P(\cdot,z)} h \tag{3.3.71}$$

Lemma 3.3.18 *Assume P is Lipschitz and satisfies (3.3.87).*

The set $\mathcal{Z} = \{z \in [0,H] : 0 \in Argmin\Pi_P(\cdot,z)\}$ when non empty is a closed interval of the form $[0, z^*]$. In the case $\mathcal{Z}$ is empty we set $z^* = 0$.

Proof: Assume $\mathcal{Z}$ is non empty and set z^* to be its supremum. By definition of z^*, $\mathcal{Z} \subset [0, z^*]$. Conversely, let z_n be a sequence in $\mathcal{Z}_0$ such that z_n converges to z^*. Then we use lemma 3.3.5 (ii) to obtain that $0 \in Argmin\Pi_P(\cdot, z^*)$, that is $z^* \in \mathcal{Z}$. Lemma 3.3.5 (iii) ensures that $[0, z^*) \subset \mathcal{Z}$. $\square$

Lemma 3.3.19 *Let z^* be as in lemma 3.3.18. Assume P is Lipschitz and satisfies (3.3.87).*

(i) There exists $c_0 > 0$ such that if $z^* \leq z_1 \leq z_2 \leq H$ and $h_i \in Argmin\Pi_P(\cdot, z_i)$ $i = 1, 2$, then

$$z_2 - z_1 \leq c_0(h_2 - h_1) \tag{3.3.72}$$

(ii) For any $z_1, z_2 \in [z^*, H]$, $Argmin\Pi_P(\cdot, z_1) \cap Argmin\Pi_P(\cdot, z_2) = \emptyset$ if $z_1 \neq z_2$

(iii) h^- , h^+ are strictly increasing on $[z^*, H]$.

Proof: Let $m(s) = \frac{\Omega^2 r_0^2}{2(1-2sr_0^2)}$. We check that m is lipschitz continuous on $[0, M_P]$. Here, M_P is defined as in lemma 3.3.3.

Set

$$\alpha(s, z) = \frac{\Omega^2 r_0^2}{2(1 - 2sr_0^2)} - P(s, z)$$

As P satisfies the first equation in (3.3.15), we have

$$-R_0 \leq \partial_s \alpha(s, z) \leq Lip(m) \tag{3.3.73}$$

Let $z_1, z_2 \in (z^*, H]$ such that $z_1 < z_2$ and $h_i \in Argmin\Pi_P(\cdot, z_i)$ $i = 1, 2$. Remark 3.3.6 ensures that $\alpha(h_2, z_2) = \alpha(h_1, z_1) = 0$ and so

$$\alpha(h_2, z_1) - \alpha(h_2, z_2) = \alpha(h_2, z_1) - \alpha(h_1, z_1). \tag{3.3.74}$$

We exploit the second equation in (3.3.87) to obtain that

$$\alpha(h_2, z_1) - \alpha(h_2, z_2) = \int_{z_2}^{z_1} \partial_z \alpha(h_2, z) dz = \int_{z_1}^{z_2} \partial_z P(h_2, z) dz \geq \frac{1}{R_0}(z_2 - z_1). \tag{3.3.75}$$

The second inequality in (3.3.73) leads to

$$\alpha(h_2, z_1) - \alpha(h_1, z_1) = \int_{h_1}^{h_2} \partial_s \alpha(s, z_1) ds \leq Lip(m)(h_2 - h_1). \tag{3.3.76}$$

We combine (3.3.74- 3.3.76) to conclude that

$$(z_2 - z_1) \leq R_0 Lip(m)(h_2 - h_1) = c_0(h_2 - h_1).$$

for all $z^* < z_1 \leq z_2 \leq H$. Note that if $\mathcal{Z} = \emptyset$, the argument above is valid when $z_1 = z^* = 0$. In the sequel, we assume that $\mathcal{Z} \neq \emptyset$. To obtain the inequality (3.3.72) when $z_1 = z^*$, we consider a sequence $\{\bar{z}^n\}_{n=1}^{\infty}$ in $(z^*, H]$ such that $\bar{z}^n > z^*$ and $\{\bar{z}^n\}_{n=1}^{\infty}$ converges to z^*. Let $h^n \in Argmin\Pi_P(\cdot, \bar{z}^n)$. As $\{\bar{z}^n\}_{n=1}^{\infty}$ converges to z^*, lemma 3.3.5 (ii) ensures that $\{h^n\}_{n=1}^{\infty}$ converges to $\{0\} = Argmin\Pi_P(\cdot, z^*)$. We let n go to ∞ in $z_2 - \bar{z}^n \leq c_0(h_2 - h^n)$ to obtain the desired result. (ii) and (iii) follow directly from (3.3.72). □

We recall that

$$h^+(z) = \max_{Argmin\Pi_P(\cdot,z)} h, \qquad h^-(z) = \min_{Argmin\Pi_P(\cdot,z)} h$$

We define

$$\mathfrak{a}(s) = \inf\Big\{z \in [z^*, H] : h^-(z) \geq s\Big\} \tag{3.3.77}$$

Lemma 3.3.20 *Let z^* be as in lemma 3.3.18 such that $z^* < H$. Assume P satisfies (3.3.87)*

(i) If $s \in (h^+(z^*), h^-(H))$ then $\mathfrak{a}(s)$ is an interior point in (z^*, H).

(ii) If $s \in [h^-(z^*), h^+(H)]$ then $s \in [h^-(\mathfrak{a}(s)), h^+(\mathfrak{a}(s))]$. As a consequence the disjoint union of $\{[h^-(z), h^+(z)]\}_{z^* \leq z \leq H}$ covers $[h^-(z^*), h^-(H)]$. Moreover, if $s \in [h^-(z), h^+(z)]$ for some $z \in [z^*, H)$ then $\mathfrak{a}(s) = z$.

(iii) $\mathfrak{a}$ is non decreasing on $\in [0, h^-(H)]$.

Proof: Let $s \in (h^+(z^*), h^-(H))$ and set $A(s) = \{z \in [z^*, H] : h^-(z) \geq s\}$ so that

$$\mathfrak{a}(s) = \inf A(s) \tag{3.3.78}$$

We note that $H \in A(s)$. Let $\mathfrak{a}(s) < z \leq H$, by (3.3.78) there exists $\bar{z} \in A(s)$ such that $\mathfrak{a}(s) < \bar{z} < z$. Consequently, $h^-(\bar{z}) \geq s$ and as h^- increasing, $h^-(\bar{z}) \leq h^-(z)$. We conclude that $h^-(z) \geq s$ and so $z \in A(s)$. Hence $(\mathfrak{a}(s), H] \subset A(s)$. We next show that $\mathfrak{a}(s)$ is an interior point of the interval $[z^*, H]$.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in (z^*, H) such that $\{a_n\}_{n=1}^{\infty}$ converges to z^*. We use the right continuity of h^+ (cfr lemma 3.3.7 iv) to obtain that $\{h^+(a_n)\}_{n=1}^{\infty}$ converges to $h^+(z^*)$. As $s > h^+(z^*)$ we obtain

$$s > h^+(a_n) > h^+(z^*) \tag{3.3.79}$$

for n big enough. We next choose a_n in (3.3.79) to be points of continuity of h^- so that $h^+(a_n) = h^-(a_n)$(cfr lemma 3.3.7 v). Therefore (3.3.79) becomes

$$s > h^-(a_n) > h^+(z^*) \tag{3.3.80}$$

for $n \geq n_0$ for some $n_0 \in \mathbb{N}$. In light of the defintion of $A(s)$, the first inequality in (3.3.80) implies that $a_n \in (z^*, H) \setminus A(s)$ and so in view of (3.3.78) $a_n \leq \mathfrak{a}(s)$ for all $n \geq n_0$. Since a_n converges to z^*, there exists $p_0 > n_0$ such that $a_{p_0} < \mathfrak{a}(s)$. The second inequality in (3.3.80) implies that $h^-(a_{p_0}) > h^+(z^*)$. This combined with $h^+(z^*) \geq h^-(z^*)$ gives $h^-(a_{p_0}) > h^-(z^*)$. By lemma 3.3.18 h^- is strictly increasing on $[z^*, H]$ and so $a_{p_0} > z^*$. We conclude that

$$z^* < \mathfrak{a}(s) \tag{3.3.81}$$

Set $b_n = H - \frac{1}{n}$. By the left continuity of h^-(cfr lemma 3.3.7 iv), $\{h^-(b_n)\}_{n=1}^{\infty}$ converges to $h^-(H)$. This, with the fact that $s < h^-(H)$ yields

$$s < h^-(b_n) < h^-(H) \tag{3.3.82}$$

for n big enough. For such n, $b_n \in A(s)$ so that $\mathfrak{a}(s) \leq b_n$. This, combined with $b_n < H$, yields

$$\mathfrak{a}(s) < H \tag{3.3.83}$$

From (3.3.81) and (3.3.83) we conclude that $\mathfrak{a}(s) \in (z^*, H)$ which proves (i).

Let $s \in (h^+(z^*), h^-(H))$. Since $\mathfrak{a}(s)$ is an interior point in the interval $[z^*, H]$, there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in (z^*, H) such that $\mathfrak{a}(s) < z_n$ and $\{z_n\}_{n=1}^{\infty}$ converges to $\mathfrak{a}(s)$. As $(\mathfrak{a}(s), H] \subset A(s)$, we have that $z_n \in A(s)$ and so $h^-(z_n) \geq s$. Without loss of generality take z_n to be a point of continuity of h^- so that $h^-(z_n) = h^+(z_n)$. Therefore, as h^+ is right continuous, $h^+(\mathfrak{a}(s)) = \lim_{n\to\infty} h^+(z_n) = \lim_{n\to\infty} h^-(z_n) \geq s$. On the other hand let $\{\bar{z}_n\}_{n=1}^{\infty}$ be a sequence in (z^*, H) such that $\{\bar{z}_n\}_{n=1}^{\infty}$ converges to $\mathfrak{a}(s)$ and $\bar{z}_n < \mathfrak{a}(s)$ so that $\bar{z}_n \notin A(s)$. Then, necessarily $h^-(\bar{z}_n) < s$. Hence $h^-(\mathfrak{a}(s)) = \lim_{n\to\infty} h^-(\bar{z}_n) \leq s$ by using the left continuity of h^-. We conclude that

$$s \in [h^-(\mathfrak{a}(s)), h^+(\mathfrak{a}(s))]. \tag{3.3.84}$$

Thus

$$(h^+(z^*), h^-(H)) \subset \bigcup_{h^+(z^*)<s<h^-(H)} [h^-(\mathfrak{a}(s)), h^+(\mathfrak{a}(s))] \tag{3.3.85}$$

As $\mathfrak{a}(s) \in [z^*, H]$ and

$$[h^-(z^*), h^+(H)] = [h^-(z^*), h^+(z^*)] \cup (h^+(z^*), h^-(H)) \cup [h^-(H), h^+(H)]$$

we obtain

$$[h^+(z^*), h^-(H)] \subset \bigcup_{z^*\leq z\leq H} [h^-(z), h^+(z)] \qquad \text{(disjoint union)} \tag{3.3.86}$$

lemma 3.3.19 ensures that the family $\{[h^-(z), h^+(z)]\}_{z^*\leq z\leq H}$ is disjoint.

Let $s \in [h^-(z), h^+(z)]$ for some $z \in (z^*, H)$. By (3.3.84), $s \in [h^-(\mathfrak{a}(s)), h^+(\mathfrak{a}(s))]$. In view of the fact that sets in (3.3.86) are disjoint, we have $h^-(z) = h^-(\mathfrak{a}(s))$ and the strict monotonicity of h^- ensures that $z = \mathfrak{a}(s)$. when $z = z^*$, we have $\mathfrak{a}(s) = z^*$ for all $s \in [h^-(z^*), h^-(z^*)]$ by using (3.3.77). Thus, if $s \in [h^-(z^*), h^-(z^*)]$ for some $z \in [z^*, H)$, then $\mathfrak{a}(s) = z$.

(b) Let $s_i \in [h^-(z^*), h^-(H)] \quad i = 1, 2$. Then $s_i \in [h^-(\mathfrak{a}(s_i)), h^+(\mathfrak{a}(s_i))]$. Since the family $\{[h(z), h^-(z)]\}_{z^* \leq z \leq H}$ is disjoint, either the intervals $[h^-(\mathfrak{a}(s_1)), h^+(\mathfrak{a}(s_1))]$ and $[h^-(\mathfrak{a}(s_2)), h^+(\mathfrak{a}(s_2))]$ are the same or they are disjoint. Assume $s_1 < s_2$. As $s_i \in [h^-(\mathfrak{a}(s_i)), h^+(\mathfrak{a}(s_i))]$, in the case where the 2 intervals above are disjoint, we necessarily have $h^-(\mathfrak{a}(s_1)) < h^-(\mathfrak{a}(s_2))$. In light of the fact that h^- is strictly increasing (and thus injective), we conclude that in all cases either $\mathfrak{a}(s_1) = \mathfrak{a}(s_2)$ or $\mathfrak{a}(s_1) < \mathfrak{a}(s_2)$. We have thus establised that $\mathfrak{a}$ is increasing on $[h^-(z^*), h^-(H)]$. If $h^-(z^*) > 0$ by definition of z^* we necessarily have $z^* = 0$. In this case, we easily check that $\mathfrak{a}(s) = 0$ for all $s \in [0, h^-(z^*))$ by using (3.3.77) and as $\mathfrak{a}$ has values in $[0.H]$ we conclude that $\mathfrak{a}$ is increasing on $[0, h^-(H)]$ $\square$

$$0 \leq \frac{\partial P}{\partial s}(s, z) \leq R_0 \qquad \text{and} \qquad \frac{1}{R_0} \leq \frac{\partial P}{\partial z}(s, z) \leq R_0 \tag{3.3.87}$$

In this case $Argmin \Pi_P(\cdot, z)$ is compact. We recall that

$$h^+(z) = \max_{Argmin \Pi_P(\cdot, z)} h, \qquad h^-(z) = \min_{Argmin \Pi_P(\cdot, z)} h \tag{3.3.88}$$

Corollary 3.3.21 *Assume the hypotheses in lemma 3.3.20 hold. The function $\mathfrak{a}$ is lipschitz continuous.*

Proof: We first note that as $\mathfrak{a}$ is non decreasing, we only need to show that

$$\mathfrak{a}(s_2) - \mathfrak{a}(s_1) \leq c_0(s_2 - s_1) \tag{3.3.89}$$

for all $s_2 \geq s_1$ in $[0, h^-(H)]$ and some constant c_0.

(a) Assume $h^+(z^*) < s_1 < h^-(H)$. Lemma 3.3.20 (ii) on $[z^*, H]$ ensures that $s_1 \leq h^+(\mathfrak{a}(s_1))$ so that $h^+(z^*) < h^+(\mathfrak{a}(s_1))$. As h^+ is strictly increasing on $[z^*, H]$ (see lemma 3.3.19), we obtain that $z^* < \mathfrak{a}(s_1)$. Let $s_2 \in [0, h^-(H)]$ such that $s_1 < s_2$. As $\mathfrak{a}$ is increasing $\mathfrak{a}(s_1) \leq \mathfrak{a}(s_2)$. Thus, $z^* < \mathfrak{a}(s_1) \leq \mathfrak{a}(s_2)$. If $\mathfrak{a}(s_1) = \mathfrak{a}(s_2)$ then (3.3.89) holds. In the sequel, we assume $z^* < \mathfrak{a}(s_1) < \mathfrak{a}(s_2)$. Choose $\bar{z}^n > \mathfrak{a}(s_1)$ such that $\{\bar{z}^n\}_{n=1}^{\infty}$ converges to $\mathfrak{a}(s_1)$ and $\bar{z}^n$ are points of continuity of h^-, that is, $h^-(\bar{z}^n) = h^+(\bar{z}^n)$. We use the fact that h^+ is non decreasing to obtain

$$h^+(\mathfrak{a}(s_1)) \leq h^+(\bar{z}^n) = h^-(\bar{z}^n).$$

This, with the fact that $s_1 \leq h^+(\mathfrak{a}(s_1))$ implies that $s_1 \leq h^-(\bar{z}^n)$ which we use along with (3.3.72) and the fact that $h^-(\mathfrak{a}(s_2)) \leq s_2$ (see lemma 3.3.20 (ii)) to get

$$\mathfrak{a}(s_2) - \bar{z}^n \leq c_0(h^-(\mathfrak{a}(s_2)) - h^-(\bar{z}^n)) \leq c_0(s_2 - s_1)$$

By letting $n \to \infty$ we obtain (3.3.89) for $h^+(z^*) < s_1 < s_2 \leq h^-(H)$.

By lemma 3.3.20 (ii), $\mathfrak{a}(s) = z^*$ for all $s \in [h^-(z^*), h^+(z^*)]$. To show that $\mathfrak{a}$ is lipschitz continuous on $[h^-(z^*), h^-(H)]$, it suffices to show that $\mathfrak{a}$ is continuous at $h^+(z^*)$ and more precisely right continuous at $h^+(z^*)$.

Let $h^+(z^*) \leq s_n$ such that $\{s_n\}_{n=1}^{\infty}$ converges to $h^+(z^*)$. By lemma 3.3.20 (ii) $s_n \in [h^-(\mathfrak{a}(s_n)), h^+(\mathfrak{a}(s_n))]$ so that $h^-(\mathfrak{a}(s_n))$ converges to $h^+(z^*)$. We use (3.3.72) to obtain that

$$0 \leq \mathfrak{a}(s_n) - z^* \leq c_0(h^-(\mathfrak{a}(s_n)) - h^+(z^*)) \tag{3.3.90}$$

As $h^-(\mathfrak{a}(s_n))$ converges to $h^+(z^*)$, (3.3.90) implies that $\mathfrak{a}(s_n)$ converges to z^*. We conclude that $\mathfrak{a}$ is continuous at $h^+(z^*)$ and so lipschitz on $[h^-(z^*), h^-(H)]$. In the case where $h^-(z) > 0$, we have $z^* = 0$ by definition of z^*. But $\mathfrak{a}(s) = 0$ on $[0, h^-(z^*)]$ Therefore, the result holds on $[0, h^-(H)]$. . $\square$

Set

$$\mathfrak{D}_{h^-} = \big\{(s, z) : z^* \leq z \leq H, 0 \leq s \leq h^-(z)\big\}$$

$$Q = \left\{(s,z) : z^* < z < H, 0 < s < h^-(H), \;\; z^* < \mathfrak{a}(s) < z\right\}$$

$$Q^* = \left\{(s,z) : z^* \leq z \leq H, 0 \leq s \leq h^-(H), \;\; z^* \leq \mathfrak{a}(s) \leq z\right\}$$

Lemma 3.3.22 *Assume the hypotheses in lemma 3.3.20 hold. Then $Q \subset \mathfrak{D}_{h^-} \subset Q^*$ .Q is open and Q^* is closed.*

Proof: 1. $Q \subset \mathfrak{D}_{h^-} \subset Q^*$

Let $(\bar{s}, \bar{z}) \in Q$. To show that $(\bar{s}, \bar{z}) \in \mathfrak{D}_{h^-}$, we only need to show that $\bar{s} \leq h^-(\bar{z})$. Note that $z^* < \bar{z} \leq H$ and as h^- is increasing, we have $h^-(z^*) \leq h^-(\bar{z})$. Assume $h^-(z^*) > 0$ (in this case $z^* = 0$). If $0 < \bar{s} < h^-(z^*)$ then $\bar{s} \leq h^-(\bar{z})$ and so $(\bar{s}, \bar{z}) \in \mathfrak{D}_{h^-}$. In the sequel, we assume $h^-(z^*) \leq \bar{s} < h^-(H)$. As $z^* < \mathfrak{a}(\bar{s}) < \bar{z}$, by lemma 3.3.18 (ii) $[h^-(\mathfrak{a}(\bar{s}), h^+(\mathfrak{a}(\bar{s})]$ and $[h^-(\bar{z}), h^+(\bar{z})]$ are disjoint. This, combined with the fact that h^- is increasing guarantees $h^+(\mathfrak{a}(\bar{s})) \leq h^-(\bar{z})$. But lemma 3.3.20 (ii) again ensures that $\bar{s} \leq h^+(\mathfrak{a}(\bar{s}))$. We conclude that $\bar{s} \leq h^-(\bar{z})$. Hence $Q \subset \mathfrak{D}_{h^-}$.

Let $(\bar{s}, \bar{z}) \in \mathfrak{D}_{h^-}$. To obtain that $\mathfrak{D}_{h^-} \subset Q^*$, it suffices to show that $\mathfrak{a}(\bar{s}) \leq \bar{z}$. As $\bar{s} \leq h^-(\bar{z})$, we use the fact that $\mathfrak{a}$ is non decreasing to obtain that $\mathfrak{a}(\bar{s}) \leq \mathfrak{a}(h^-(\bar{z}))$. by lemma 3.3.20 (ii), $\mathfrak{a}(h^-(\bar{z})) = \bar{z}$ so that $\mathfrak{a}(\bar{s}) \leq \bar{z}$. Hence $(\bar{s}, \bar{z}) \in Q^*$. We conclude that $\mathfrak{D}_{h^-} \subset Q^*$.

2. We recall that $\mathfrak{a}$ is continuous (see corollary 3.3.21). Therefore the fact that Q is open and Q^* is closed is trivial.

$\square$

Set

$$Q_1 = \left\{(s,z) : 0 \leq s \leq h^-(H), \;\; z = \mathfrak{a}(s)\right\}, \; Q_2 = \left\{(s,z) : z = H, \; 0 \leq s \leq h^-(H)\right\}$$

$$Q_3 = \{(s,z) : s = 0 \;\; z^* \leq z \leq H\} \; Q_4 = \left\{(s,z) : z = z^*, \; 0 \leq s \leq h^-(z^*)\right\}$$

Lemma 3.3.23 *Assume the hypotheses in lemma 3.3.20 hold. D_{h^-} is a domain with a Lipschitz boundary.*

Proof: We claim that

$$\partial \mathfrak{D}_{h^-} = \bigcup_{1 \le i \le 4} Q_i$$

Indeed, we easily check that Q^* is the closure of Q and that $\partial Q = \bigcup_{1 \le i \le 4} Q_i$. The lemma 3.3.22 implies that the interior and the closure of the domain $\mathfrak{D}_{h^-}$ are respectively Q and Q^*. This proves the claim. As Q_i are graphs of lipschitz functions, $\mathfrak{D}_{h^-}$ has a Lipschitz boundary. But

$$D_{h^-} = \mathfrak{D}_{h^-} \cup \{(s,z) : s = 0,\ 0 \le z \le z^*\}$$

The result follows immediately. □

3.3.5 Existence and uniqueness of a solution in the Monge -Ampere equation.

Theorem 3.3.24 *Let $R_0 > 0$. Let σ be a probability measure on $\mathbb{R}^2$ such that the support of σ is contained in $[0, R_0]^2$. Then (3.3.1) admits a unique solution $(\bar{\Psi}, \bar{P}, \bar{h})$. $(\bar{\Psi}, \bar{P})$ is obtained as the maximizer in (3.3.11) and $\bar{h}$ is monotone and obtained as the minimiser in (3.3.5). Moreover, if the support of σ is contained in $[\frac{1}{R_0}, R_0] \times [0, R_0]$ then $\partial D_{\bar{h}}$ is Lipschitz continuous.*

Proof: Proposition 3.3.17 shows that (3.3.11) has a unique maximizer $(\bar{\Psi}, \bar{P})$ and (3.3.5) has a unique minimizer $\bar{h}$ so that (3.3.1) has a unique solution. As $e(s)\chi_{D_{\bar{h}}}$ is a probability measure and $\bar{h}$ monotone non decreasing , $\{\bar{h} > 0\}$ is of positive Lebesgue measure so that $z* < H$ (z^* is as defined in lemma 3.3.18). Thus, by lemma 3.3.23, if the support of σ is contained in $[\frac{1}{R_0}, R_0] \times [0, R_0]$ then $\partial D_{\bar{h}}$ is Lipschitz continuous.
□

3.4 Some stability results.

Theorem 3.3.24 generates two operators $\mathcal{H}$, $\bar{\mathcal{H}}$ defined in the following way: To any $\sigma \in \mathcal{P}([0, R_0]^2)$, we associate $h = \mathcal{H}(\sigma)$ the minimizer in (3.3.5) and $(P, \Psi) = \bar{\mathcal{H}}(\sigma)$ the maximizer in (3.3.11).

Remark 3.4.1 *Let* $\sigma \in \mathcal{P}([0, R_0]^2)$. *If* $h = \mathcal{H}(\sigma)$, $(P, \Psi) = \bar{\mathcal{H}}(\sigma)$ *and* $\gamma = (\nabla P \times \mathbf{id})_{\#} e(s)\chi_{D_h}$ *then Propositon 3.3.17 and (3.3.5) yield that*

$$\bar{I}[\sigma](h) = I(h, \gamma) = J[\sigma](\Psi, P)$$

and

$$\gamma \in \Gamma_0(\sigma, e(s)\chi_{D_h})$$

Lemma 3.4.2 *Let* $\{\sigma_n\}_{n=1}^{\infty}$ *and* σ *be probability measures on* $\mathcal{P}([0, R_0]^2)$ *such that* $\{\sigma_n\}_{n=1}^{\infty}$ *converges to* σ *narrowly. Let* $(P_n, \Psi_n) = \bar{\mathcal{H}}(\sigma_n)$ *then* $\{P_n\}_{n=1}^{\infty}$ *and* $\{\Psi_n\}_{n=1}^{\infty}$ *are precompact respectively in* $C(\bar{\Delta}_{r_0})$ *and* $C([0, R_0]^2)$.

Proof: Let $h_n = \mathcal{H}(\sigma_n)$. Set $\bar{P}_n = P_n - \lambda_n$, $\bar{\Psi}_n = \Psi_n + \lambda_n$ and $\lambda_n = P_n(0, 0)$. By using lemma 3.1.2 first and then Proposition 3.3.17 (iii), we obtain a constant $c_0 \in \mathbb{R}$ such that

$$c_0 \leq \bar{I}[\sigma_n](h_n) = J[\sigma_n](P_n, \Psi_n) = J[\sigma_n](\bar{P}_n + \lambda_n, \Psi_n - \lambda_n)$$

Therefore , in view of lemma 3.3.12, $|\lambda_n| \leq C_0$. In light of remark 3.3.11, we obtain that $\{\bar{P}_n\}_{n=1}^{\infty}$ and $\{\bar{\Psi}_n\}_{n=1}^{\infty}$ are uniformly lipschitz and uniformly bounded. This, combined with the fact that $|\lambda_n| \leq C_0$ implies that $\{P_n\}_{n=1}^{\infty}$ and $\{\Psi_n\}_{n=1}^{\infty}$ are uniformly lipschitz and uniformly bounded as well. Thus, by Arzela- Ascoli, $\{P_n\}_{n=1}^{\infty}$ and $\{\Psi_n\}_{n=1}^{\infty}$ are precompact respectively in $C(\bar{\Delta}_{r_0})$ and $C([0, R_0]^2)$. $\square$

Recall

$$\bar{I}[\sigma](h) = \frac{1}{2} W_2^2(\sigma, e(s)\chi_{D_h}) + \int_{\mathbb{R}^2} f(p)\sigma(dp) + \int g(q)e(s)\chi_{D_h}(dq).$$

where

$$f(p) := \frac{\Omega^2 r_0^2}{2(1 - 2sr_0^2)} - \frac{|p|^2}{2} \text{ and } g(q) := \frac{\Upsilon}{2r_0^2} - \Omega\sqrt{\Upsilon} - \frac{|q|^2}{2}$$

and

$$\mathcal{H}_{dom} = \left\{ h : [0, H] \longrightarrow [0, 1/(2r_0^2)) \mid \int_{\mathbb{R}^2} e(s)\chi_{D_h} ds dz = 1 \right\}$$

Lemma 3.4.3 *Let $\{\sigma_n\}_{n=1}^{\infty}$, σ be probability measures in $[0, R_0]^2$ and let $h_n = \mathcal{H}(\sigma_n)$, $h = \mathcal{H}(\sigma)$, $\bar{\mathcal{H}}(\sigma) = (P, \Psi)$ and $\bar{\mathcal{H}}(\sigma_n) = (P_n, \Psi_n)$ for $n \geq 1$. Assume that σ_n converges narrowly to σ. Then*

(i) $\{h_n\}_{n=1}^{\infty}$ converges pointwise to h and so $e(s)\chi_{D_{h_n}}$ converges narrowly to $e(s)\chi_{D_h}$. Moreover, if $\{P_n\}_{n=1}^{\infty}$ is uniformly convergent in $C(\bar{\Delta}_{r_0})$ then there exists $M > 0$ such that

$$2r_0^2 M < 1 \qquad and \qquad 0 \leq h_n, h < M \quad for\ n \geq 1. \tag{3.4.1}$$

(ii)

$$\nabla P_n \longmapsto \nabla P \qquad \mathcal{L}^2 - a.e\ in\ \Delta_{r_0}. \tag{3.4.2}$$

Moreover, suppose in addition that $\{\sigma_n\}_{n=1}^{\infty}$, σ are absolutely continuous with respect to Lebesgue. Then

$$\nabla \Psi_n \longmapsto \nabla \Psi \qquad \mathcal{L}^2 - a.e\ in\ \mathbb{R}^2. \tag{3.4.3}$$

Proof:

(i) Let's extract from $\{h_n\}_{n=1}^{\infty}$ a subsequence still denoted by $\{h_n\}_{n=1}^{\infty}$. By Helly's theorem, there exists a subsequence $\{h_{n_k}\}_{k=1}^{\infty}$ of $\{h_n\}_{n=1}^{\infty}$ and a monotone function $\bar{h}$ such that $\{h_{n_k}\}_{k=1}^{\infty}$ converges pointwise to $\bar{h}$. We use the minimality property of h_{n_k} and lemma 3.1.2 to obtain

$$\bar{I}[\sigma_{n_k}](h_{n_k}) \leq \bar{I}[\sigma_{n_k}](\hat{h}) \leq \frac{\Omega^2}{8} ||e \circ \hat{h}||_{L^1[0,H]} + \bar{c}_0 \tag{3.4.4}$$

for all $\hat{h} \in \mathcal{H}_{dom}$. Set $\hat{h} = h_{00}$ (h_{00} is as defined in remark 3.1.1) , corollary 3.1.4 ensures that $\bar{I}[\sigma_{n_k}](h_{n_k})$ converges to $\bar{I}[\sigma](\bar{h})$. The same corollary guarantees that $\bar{I}[\sigma_{n_k}](\hat{h})$ converges to $I[\sigma](\hat{h})$. Therefore, (3.4.4) becomes in the limit

$$\bar{I}[\sigma](\bar{h}) \leq \bar{I}[\sigma](\hat{h})$$

As $\hat{h}$ is arbitrary in $\mathcal{H}_{dom}$, we conclude that $\bar{h} = \mathcal{H}(\sigma)$. The uniqueness of this minimizer ensures that $h = \bar{h}$, $a.e.$ Thus $\{h_{n_k}\}_{k=1}^{\infty}$ converges to h. We conclude that the whole sequence $\{h_n\}_{n=1}^{\infty}$ converges to h. Remark 3.3.10 provides M such that (3.4.1) holds.

(ii) Let $\gamma_n \in \Gamma_0(\sigma_n, e(s)\chi_{D_{h_n}})$ and $\gamma \in \Gamma_0(\sigma, e(s)\chi_{D_h})$. By [[1], Proposition 7.1.3], every subsequence of $\{\gamma_n\}_{n=1}^{\infty}$ is relatively compact in $\mathcal{P}(\Delta_{r_0} \times [0, R_0]^2)$ and its limit point belongs to $\Gamma_0(\sigma, e(s)\chi_{D_h})$. As $\Gamma_0(\sigma, e(s)\chi_{D_h})$ reduces to $\{\gamma\}$, we conclude that $\{\gamma_n\}_{n=1}^{\infty}$ converges to γ. We have the following representation of γ_n and γ :

$$\gamma_n = (\nabla P_n \times \mathbf{id})_{\#} e(s)\chi_{D_{h_n}} \qquad \gamma = (\nabla P \times \mathbf{id})_{\#} e(s)\chi_{D_h}$$

so that $spt(\gamma_n) \subset \partial_{.} P_n \subset \Delta_{r_0} \times [0, R_0]^2$ and $spt(\gamma) \subset \partial_{.} P \subset \Delta_{r_0} \times [0, R_0]^2$. We consider a subsequence of $\{P_n\}_{n=1}^{\infty}$ and $\{\Psi_n\}_{n=1}^{\infty}$ still denoted respectively by $\{P_n\}_{n=1}^{\infty}$ and $\{\Psi_n\}_{n=1}^{\infty}$ for simplicity. By lemma 3.4.2 , up to a subsequence $\{(P_n, \Psi_n)\}_{n=1}^{\infty}$ converges locally uniformly to some $(\bar{P}, \bar{\Psi})$. Note that as $P_n,\ \Psi_n$ are convex, $\bar{P}\ \bar{\Psi}$ are convex. The convexity of the P_n implies that

$$P_n(p) \geq P_n(\bar{p}) + < \nabla P_n(\bar{p}), p - \bar{p} > \tag{3.4.5}$$

for all $p \in \Delta_{r_0}$ and any point $\bar{p}$ where $\{P_n\}_{n=1}^{\infty}$ are differentiable. Fix $\bar{p}_0 \in \Delta_{r_0}$ such that $\{P_n\}_{n=1}^{\infty}$ and $\bar{P}$ are all differentiable at $\bar{p}_0$ so that $\partial_{.} P_n(\bar{p}_0) = \{\nabla P_n(\bar{p}_0)\}$ and $\partial_{.} \bar{P}(\bar{p}_0) = \left\{\nabla \bar{P}(\bar{p}_0)\right\}$. Note that $\{\nabla P_n(\bar{p}_0)\}_1^{\infty} \subset [0, R_0] \times [0, R_0]$. Then there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of integers such that $\{\nabla P_{n_k}(\bar{p}_0)\}_{k=1}^{\infty}$ converges. By passing to the limit in (3.4.5) when $\bar{p}$ is replaced by $\bar{p}_0$ we obtain that $\{\nabla P_{n_k}(\bar{p}_0)\}_{k=1}^{\infty}$ converges to $\nabla \bar{P}(\bar{p}_0)$. Since the set of points of differentiability of P_n and P is $\mathcal{L}^2$ negligible, we eventually get

$$\{\nabla P_{n_k}\}_{k=1}^{\infty} \text{ converges to } \nabla \bar{P} \qquad \text{a.e-}\mathcal{L}^2. \tag{3.4.6}$$

To obtain (3.4.2), we only need to show $\nabla P = \nabla \bar{P}$ a.e-$\mathcal{L}^2$ so that the limit in (3.4.6) is independent the subsequence considered. Let $(a, b) \in spt(\gamma) \subset \partial P$. Since $\{\gamma_{n_k}\}_1^{\infty}$ narrowly converges to γ, there exists $(a_k, b_k) \in spt(\gamma_{n_k}) \subset \partial_{.} P_{n_k}$ such that (a_k, b_k) converges to (a, b). The subdifferential inequality for the convex functions P_{n_k}

$$P_{n_k}(p) \geq P_{n_k}(a_k) + < b_k, p - a_k >$$

This yields in the limit

$$\bar{P}(p) \geq \bar{P}(a) + < b, p - a >$$

Therefore $(a, b) \in \partial \bar{P}$. If $\bar{P}$ and P are differentiable at a then $\nabla P(a) = \nabla \bar{P}(a)$. As the points of non-differentiability of $\bar{P}$ and P are negligible with respect to Lebesgue, we conclude that $\nabla P = \nabla \bar{P}$ a.e-$\mathcal{L}^2$.

(b) Suppose now that $\{\sigma_n\}_{n=1}^{\infty}$ and σ are absolutely continuous with respect to Lebesgue, we have

$$\gamma_n := (\mathbf{id} \times \nabla \Psi_n)_{\#} \sigma_n, \qquad \gamma := (\mathbf{id} \times \nabla \Psi)_{\#} \sigma$$

so that $spt(\gamma_n) \subset \partial \Psi_n$ and $spt(\gamma) \subset \partial \Psi$.

We assume without loss of generality that (P_n, Ψ_n) converge uniformly in $C(\bar{\Delta}_{r_0}) \times C([0, R_0]^2)$. By (i), $\{h_n\}_{n=1}^{\infty}$ and h are uniformy bounded above by some $0 < M < \frac{1}{2r_0^2}$ so that $e(s)\chi_{D_{h_n}}$ and $e(s)\chi_{D_h}$ are supported in $[0, M] \times [0, H] \subset \Delta_{r_0}$. By a reasoning similar to the one in Part (a), we obtain (3.4.3).

CHAPTER IV

A TOY MODEL OF THE ALMOST AXISYMMETRIC FLOWS WITH FORCING TERMS.

In this Chapter we consider a toy model related to the Almost axisymmetric Flows with Forcing Terms by reducing the problem to a 2-dimensional system of PDEs. Though the resulting simplified version may not be physical, we hope to gain some insights into the structure of the full system. The unknown here are $\bar{u}, \bar{v}, \bar{w}, \bar{\theta}$ and $\bar{\varphi}$ the material derivative takes the special form $\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{v}\frac{\partial}{\partial r} + \bar{w}\frac{\partial}{\partial z}$.

$$\begin{cases} \dfrac{\mathrm{D}\bar{u}}{\mathrm{D}t} + \dfrac{\bar{u}\bar{v}}{r} + 2\Omega\bar{v} & = \frac{1}{r}\bar{F}(t,r,z), & (4.0.7a) \\ \dfrac{\mathrm{D}\bar{\theta}'}{\mathrm{D}t} & = \bar{S}(t,r,z), & (4.0.7b) \\ \dfrac{\bar{u}^2}{r} + 2\Omega\bar{u} & = \frac{\partial\bar{\varphi}}{\partial r}, & (4.0.7c) \\ \dfrac{\partial\bar{\varphi}}{\partial z} - g\dfrac{\bar{\theta}'}{\theta_0} & =0 & (4.0.7d) \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\bar{v}) + \dfrac{\partial\bar{w}}{\partial z} & =0, & (4.0.7e) \end{cases}$$

The above equations are to be solved in the moving domain

$$\Gamma_{\bar{\varsigma}_t} = \{(r,z): \quad 0 \le z \le H, \quad r_0 \le r \le \varsigma(t,z)\} \tag{4.0.8}$$

where ς is a free boundary and r_0, H are positif real numbers. The conditions on the boundary are given by

$$\begin{cases} \langle(\bar{v}_t, \bar{w}_t); \mathbf{n_t}\rangle = 0 & \text{on } \{r = r_0\} \cup \{z = 0\} \cup \{z = H\} \\ \frac{\partial\bar{\varsigma}_t}{\partial t} + \bar{w}\frac{\partial\bar{\varsigma}_t}{\partial z} = \bar{v} & on \;\; \{r = \bar{\varsigma}(t,z)\} \end{cases} \tag{4.0.9}$$

along with

$$\bar{\varphi}(t, \varsigma(t,H), H) = 0.$$

Here $\mathbf{n_t}$ is the unit outward normal vector field at time t and

$\bar{F}(t, r, z)$ and $\bar{S}(t, r, z)$ are prescribed functions. The system described above turns out to be linked to the continuity equation (4.0.11). We show that when enough regularity is assumed and $\bar{\varphi} + \frac{\Omega^2 r^2}{2}$ is strictly convex, (4.0.11) provides a solution to (4.0.7) and (4.0.9).

These equations are supplemented by the initial conditions

$$(u, v, w)_{|t=0} = (u_0, v_0, w_0);\ \theta'_{|t=0} = \theta'_0\ \ \varphi_{|t=0} = \varphi_0$$

and we initially require the conditions:

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}(rv_0) + \frac{\partial w_0}{\partial z} = 0 \\ \nabla^2\left(\varphi_0 + \Omega^2 \frac{r^2}{2}\right) > 0 \\ \frac{u_0^2}{r} + 2\Omega u_0 = \frac{\partial \varphi_0}{\partial r} \\ \frac{\partial \varphi_0}{\partial z} - g\frac{\theta'_0}{\theta_0} = 0 \end{cases} \tag{4.0.10}$$

Our goal in this chapter is to solve the following continuity equation under two different assumptions on the initial data.

$$\begin{cases} \frac{\partial \sigma}{\partial t} + \operatorname{div}(\sigma \mathbf{X}) = 0 \qquad (0, \infty) \times [0, R_0]^2 \\ \sigma_{|t=0} = \bar{\sigma}_0 \end{cases} \tag{4.0.11}$$

holds in the sense of distribution with

$$\mathbf{X}_t = \left(2\sqrt{\Upsilon} F_t\left(\frac{r_0}{\sqrt{1-2r_0^2\frac{\partial\Psi}{\partial\Upsilon}}}, \frac{\partial\Psi}{\partial Z}\right),\ \frac{g}{\theta_0} S_t\left(\frac{r_0}{\sqrt{1-2r_0^2\frac{\partial\Psi}{\partial\Upsilon}}}, \frac{\partial\Psi}{\partial Z}\right)\right) \tag{4.0.12}$$

Here $\Psi_t : \mathbb{R}^2_+ \longmapsto \mathbb{R}$ and $P_t : [0, 1/(2r_0^2)) \times [0, H] \longmapsto \mathbb{R}$ are Legendre transforms of each other and there exists $h : [0, H] \longmapsto \mathbb{R}$ satisfying

$$\begin{cases} \frac{r_0^4}{(1-2r_0^2\frac{\partial\Psi}{\partial\Upsilon})^2} \det(\nabla^2\Psi) = \sigma \\ \nabla\Psi(spt(\sigma)) = D_h \\ P(h(z), z) = \frac{\Omega^2 r_0^2}{2(1-2r_0^2 h(z))} \quad \text{on} \quad \{h > 0\} \end{cases} \tag{4.0.13}$$

In the next theorem, we show that (4.0.11) provides a solution to (4.0.7) when enough regularity is assumed. To do so, we start with the following lemma.

Lemma 4.0.4 *Let $T > 0$ and $A \subset \mathbb{R}^2$ open. Let $\{\sigma_t\}$ be a collection of probability measure and $\mathbf{X}$ a velocity field such that (4.0.11) holds. Let $\mathfrak{F}$ be a smooth function on $(0,T) \times A$ such that for $\bar{\mathfrak{F}}_t$ is invertible for all $t \in [0,T]$. Assume that there exist $\bar{v}, \bar{w}$ and $\bar{\varsigma}$ such that $\Gamma_{\bar{\varsigma}_t} \subset A$ for all $t \in [0,T]$ and*

$$\bar{\mathfrak{F}}_{t\#}(r\chi_{\Gamma_{\bar{\varsigma}_t}}) = \sigma_t \qquad \textit{and} \qquad \mathbf{X} \circ \bar{\mathfrak{F}} = \frac{\partial \bar{\mathfrak{F}}}{\partial t} + \langle (v,w), \nabla \bar{\mathfrak{F}} \rangle$$

Then

$$\begin{cases} \langle (\bar{v}_t, \bar{w}_t); \mathbf{n_t} \rangle = 0 & \textit{on } \{r = r_0\} \cup \{z = 0\} \cup \{z = H\} \\ \frac{\partial \bar{\varsigma}_t}{\partial t} + \bar{w}\frac{\partial \bar{\varsigma}_t}{\partial z} = \bar{v} & \textit{on } \{r = \bar{\varsigma}(t,z)\} \\ \frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{\partial w}{\partial z} = 0 & \textit{in } \Gamma_{\bar{\varsigma}} \end{cases}$$

Proof: The proof is similar to the proof of lemma 2.3.3. □

We recall the function $\mathbf{s} : [0,\infty) \times [0,H] \longrightarrow \Delta_{r_0} := [0, \frac{1}{2r_0^2}) \times [0,H]$ defined by

$$\mathbf{s}(r,z) = \left(\frac{1}{2}\left(r_0^{-2} - r^{-2}\right), z\right)$$

and its inverse $\mathbf{d}$ given by

$$\mathbf{d}(s,z) = \left(\frac{r_0}{\sqrt{1 - 2r_0^2 s}}, z\right)$$

Theorem 4.0.5 *Assume $P_t : \Delta_{r_0} \longmapsto \mathbb{R}$ and $\Psi_t : [0,R_0]^2 \longmapsto \mathbb{R}$ legendre transforms of each other, smooth enough and P is strictly convex so that ∇P and $\nabla \Psi$ are inverse of each other (on the interior of their domain) and P, Ψ, $\mathbf{X}$, F, and S solve (4.0.11), (4.0.12) and (4.0.13). Define $\bar{u}, \bar{\theta}'$ by*

$$(\bar{u}r + r^2\Omega)^2 = \partial_s P \circ \mathbf{s}, \qquad \frac{g}{\theta_0}\bar{\theta}' = \partial_z P \circ \mathbf{s}, \qquad \bar{\varphi} = P \circ \mathbf{s} - \frac{\Omega^2 r^2}{2} \qquad \bar{\varsigma}_t = \frac{r_0}{\sqrt{1 - 2r_0^2 h_t}} \tag{4.0.14}$$

and choose $(\bar{u}, \bar{v})$ such that

$$(\bar{u}, \bar{v}) \circ \nabla\Psi \circ \mathbf{d} = \frac{\partial}{\partial t}\nabla\Psi \circ \mathbf{d} + \langle \mathbf{X}, \nabla\Psi \circ \mathbf{d} \rangle \tag{4.0.15}$$

Then $(\bar{u}, \bar{v}, \bar{w})$, $\bar{\varphi}$, $\bar{\theta}'$ and $\bar{\varsigma}$ solve (4.0.7) and (4.0.9).

Proof: Based on the discussion in section 2.3.1, we can rewrite (4.0.15) as

$$\mathbf{X}_t \circ \nabla P \circ \mathbf{s} = \frac{\partial}{\partial t}\nabla P \circ \mathbf{s} + \langle (\bar{u}, \bar{v}), \nabla[\nabla P \circ \mathbf{s}] \rangle = \frac{D}{Dt}[\nabla P_t \circ \mathbf{s}] \tag{4.0.16}$$

Here, $\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{v}\frac{\partial}{\partial r} + \bar{w}\frac{\partial}{\partial z}$. But, by using the first two equations in (4.0.14), we obtain

$$\begin{aligned} \frac{D}{Dt}[\nabla P_t \circ \mathbf{s}] &= \left(2(\bar{u}r + \Omega r^2)\frac{D}{Dt}(\bar{u}r + \Omega r^2), \frac{g}{\theta_0}\frac{D\bar{\theta}'}{Dt} \right) \\ &= \left(2(\sqrt{\partial_s P} \circ \mathbf{s})(r\frac{\mathrm{D}\bar{u}}{\mathrm{Dt}} + \bar{u}\bar{v} + 2r\Omega\bar{v}), \frac{g}{\bar{\theta}_0}\frac{D\bar{\theta}'}{Dt} \right) \end{aligned} \tag{4.0.17}$$

and thanks to (4.0.12),

$$\mathbf{X}_t \circ \nabla P_t \circ \mathbf{s} = \left(2(\sqrt{\partial_s P_t} \circ \mathbf{s})\bar{F}_t, \frac{g}{\theta_0}\bar{S}_t \right) \tag{4.0.18}$$

We combine (4.0.16), (4.0.17) and (4.0.18) to obtain (4.0.7a) and (4.0.7b)

Thanks to remark 2.3.6, the first two equations in (4.0.14) imply that $\bar{u}$ $\bar{\varphi}$ and $\bar{\theta}'$ solve (4.0.7c) and (4.0.7d).

Note that (4.0.13) means that $\nabla\Psi$ pushes σ_t onto $e(s)\chi_{D_{h_t}}$. This implies that ∇P_t pushes $e(s)\chi_{D_{h_t}}$ onto σ_t and we easily check that $\nabla P_t \circ \mathbf{s}$ pushes $r\chi_{\Gamma_{\varsigma_t}}$ to σ_t. This, combined with (4.0.16) yields (4.0.7e) and (4.0.9) by applying lemma 4.0.4.

□

4.1 Existence of a solution for initial data that are absolutely continuous with respect to Lebesgue.

In this section, Σ denotes the set of all borel probability measures σ on $\mathbb{R}^2_+$ that are absolutely continuous with respect to Lebesgue and whose support is contained in $[0, R_0]^2$

We consider the functions $\bar{F} = \bar{F}_t(r,z), \bar{S} = \bar{S}_t(r,z)$ such that $\bar{S}, \bar{F} \in C^1((0,\infty) \times \mathbb{R}^2)$ and satisfy the following conditions:

- $(A1)$ $0 \le \bar{F}, \frac{g}{\theta_0}\bar{S} \le M$ for some positive constant M.
- $(A2)$ $\frac{\partial \bar{F}}{\partial z} = \frac{\partial \bar{S}}{\partial r} = 0$
- $(A3)$ $\frac{\partial \bar{F}}{\partial r}, \frac{\partial \bar{S}}{\partial z} > 0$

Lemma 4.1.1 *We consider a family $\sigma = \varrho\mathcal{L}^2,\ \sigma^n = \varrho^n\mathcal{L}^2 \in \mathcal{P}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ $n \ge 1$ that is equi-integrable and let $\{\mathbf{v}^n\}_{n\ge1} : \mathbb{R}^2 \longmapsto \mathbb{R}^2$ be borel measurable such that $|\mathbf{v}^n| \le M_0$ a.e where M_0 is a positive constant. Assume $\{\sigma^n\}_{n=1}^\infty$ converges narrowly to σ and $\mathbf{v}^n$ converges to $\mathbf{v}$ a.e. Then*

$$\mathbf{v}^n\sigma^n \longrightarrow \mathbf{v}\sigma \qquad \textit{in the sense of distribution.}$$

Proof: let $\phi \in C_c(\mathbb{R}^2;\ \mathbb{R}^2)$.

$$\int_{\mathbb{R}^2} \langle\phi;\ \mathbf{v}^n\rangle\varrho^n dq - \int_{\mathbb{R}^2} \langle\phi;\ \mathbf{v}\rangle\varrho dq = \int_{\mathbb{R}^2} \langle\phi;\ (\mathbf{v}^n - \mathbf{v})\rangle\varrho^n dq + \int_{\mathbb{R}^2} \langle\phi;\ \mathbf{v}\rangle(\sigma^n - \sigma)dq \quad (4.1.1)$$

As $\{\sigma^n\}_{n=1}^\infty$ converges narrowly to σ, we obtain

$$\limsup_{n\to\infty} \left| \int_{\times\mathbb{R}^2} \langle\phi;\ \mathbf{v}\rangle(\varrho^n - \varrho)dq \right| \le M_0 \limsup_{n\to\infty} \int_{\mathbb{R}^2} |\phi|(\sigma^n - \sigma)dq = 0 \qquad (4.1.2)$$

Let $\varepsilon > 0$, and denote by A_0 the support of ϕ. Since $\{\varrho^n\}_{n=1}^\infty$ are equi-integrable, there exists $\delta > 0$ such that for any Lebesgue measurable set B,

$$\mathcal{L}^2(B) \le \delta \Longrightarrow \sup_n \int_B \varrho^n dq \le \varepsilon \qquad (4.1.3)$$

As $\{\mathbf{v}^n\}_{n=1}^\infty$ converges to $\mathbf{v}$ a.e, Egoroff's theorem provides a lebesgue set $A \subset A_0$ such that $\mathcal{L}^2(A_0 \setminus A) \le \delta$ and $\mathbf{v}^n$ converges uniformly to $\mathbf{v}$ on A. Using the uniform convergence of $\{\mathbf{v}^n\}_{n=1}^\infty$, we get

$$\limsup_{n\to\infty} \left| \int_A \langle\phi;\ \mathbf{v}^n - \mathbf{v}\rangle\varrho^n dq \right| \le \|\phi\|_\infty \limsup_{n\to\infty} \|\mathbf{v}^n - \mathbf{v}\|_{L^\infty(A)} = 0 \qquad (4.1.4)$$

In view of (4.1.3) and using $|\mathbf{v}^n|, |\mathbf{v}| \leq M_0$, we obtain

$$\begin{aligned}
\limsup_{n\to\infty} |\int_{A_0\setminus A} \langle\phi;\ \mathbf{v}^n - \mathbf{v}\rangle \varrho^n dq| &\leq \limsup_{n\to\infty} \int_{A_0\setminus A} \|\phi\|_\infty |\mathbf{v}^n - \mathbf{v}|\varrho^n dq \\
&\leq 2M_0\|\phi\|_\infty \limsup_{n\to\infty} \int_{A_0\setminus A} \varrho^n dq \qquad (4.1.5)\\
&\leq 2\|\phi\|_\infty M_0 \varepsilon
\end{aligned}$$

We combine (4.1.1) (4.1.2) (4.1.4) and (4.1.5) to obtain

$$\limsup_{n\to\infty} |\int \langle\phi;\ \mathbf{v}^n\rangle \varrho^n dq - \int \langle\phi;\ \mathbf{v}\rangle \varrho dq| \leq 2M_0\varepsilon$$

As ε is arbitrary, the result is established. □

Theorem 4.1.2 *Let I be an open interval in $\mathbb{R}$ and $p \geq 1$. If a narrowly continuous curves $\sigma_t : I \longrightarrow \mathcal{P}_p(\mathbb{R}^2)$ satisfies the continuity equation*

$$\frac{\partial \sigma_t}{\partial t} + \operatorname{div}(\sigma_t \mathbf{v}_t) = 0$$

in the sense of distribution for some Borel velocity field v_t with $||\mathbf{v}_t||_{L^p(\mu_t)} \in L^1(I)$ then $\mu_t : I \longrightarrow \mathcal{P}_p(\mathbb{R}^2)$ is absolutely continuous and

$$W_p(\sigma_t, \sigma_s) \leq \int_s^t ||\mathbf{v}_\tau||_{L^p(\sigma_\tau)} d\tau \qquad (4.1.6)$$

Proof: [see [1], Page 183]

Lemma 4.1.3 *Let $a,\ \tau > 0$ and $L_a > 1$. Let $\sigma_a = \varrho_a \mathcal{L}^2$ be a borel probability measure on $\mathbb{R}^2_+$ that is absolutely continuous with respect to Lebesgue such that*

$$spt\sigma_a \subset [0, L_a]^2.$$

Assume that $\Psi : [0,\infty)\times\mathbb{R}^2_+ \longmapsto \mathbb{R}$ is such that for each $t \geq 0$ fixed, $\Psi(t,\cdot) : \mathbb{R}^2_+ \longmapsto \mathbb{R}$ is convex and whenever $\nabla\Psi(t,\cdot)$ exists, it has values in $[0, \frac{1}{2r_0^2}) \times [0, H]$. Set

$$\mathbf{v}_t(q) = \left(2\sqrt{\Upsilon}\bar{F}_t\left(\frac{r_0}{\sqrt{1-2r_0^2\frac{\partial\Psi}{\partial\Upsilon}(t,q)}}, \frac{\partial\Psi}{\partial Z}(t,q)\right),\ \frac{g}{\theta_0}\bar{S}_t\left(\frac{r_0}{\sqrt{1-2r_0^2\frac{\partial\Psi}{\partial\Upsilon}(t,q)}}, \frac{\partial\Psi}{\partial Z}(t,q)\right)\right) \qquad (4.1.7)$$

with $q = (\Upsilon, Z)$. Assume that (A_1),$(A_2)$and (A_3) hold. Then, there exists a family of measures $\sigma_t = \varrho_t \mathcal{L}^2 \in \mathcal{P}(\mathbb{R}^2_+)$ absolutely continuous with respect to Lebesgue such that

$$spt\sigma_t \subset [0, L_{a+\tau}]^2 \qquad \text{for } t \in [a, a+\tau] \text{ with } 1 < L_{a+\tau} := L_a(1+M\tau)^2$$

satisfying the following:

(a) $\int_{\mathbb{R}^2} \varrho_t^r dq \leq \int_{\mathbb{R}^2} \varrho_a^r dq$ *for any* $r \geq 1$ *and* $t \in [a, a+\tau]$.

(b) $t \longmapsto \sigma_t \in AC_1\left((a, a+\tau); \mathcal{P}(\mathbb{R}^2_+)\right)$ *and*

$$\begin{cases} \frac{\partial \sigma}{\partial t} + \operatorname{div}(\sigma \mathbf{v}_t) = 0, & (t,q) \in (a, a+\tau) \times \mathbb{R}^2 \\ \sigma_{|t=a} = \bar{\sigma}_a \end{cases} \tag{4.1.8}$$

holds in the sense of distribution.

(c) $t \longmapsto \sigma_t$ *is lipschitz continuous with respect to the* $1-$*Wasserstein distance with lipschitz constant less than* $c_0 := M\sqrt{4L_0 + 1}$ *in* $[a, a+\tau]$.

Remark 4.1.4 *Since* $\Psi(t,\cdot)$ *is convex,* $\nabla\Psi(t,\cdot)$ *exists* $\mathcal{L}^2$ *a.e so that* $\mathbf{v}_t$ *is defined* $\mathcal{L}^2$ *a.e. As* σ_t *is absolutely continuous with respect to* $\mathcal{L}^2$, $\mathbf{v}_t$ *is defined a.e* σ_t.

Proof: We subdivise the proof into several steps.

Step 1 We assume that Ψ_t is $C^2(\mathbb{R}^{*2}_+)$ for each t fixed.

We observe that the vector field $\mathbf{v}$ is smooth in $(0,\infty) \times (0,\infty)^2$ and define the associated flow by

$$\dot{\phi}_t = \mathbf{v}_t \circ \phi_t \text{ and } \phi_a = \mathbf{id} \qquad \text{for } t \in (a, a+\tau). \tag{4.1.9}$$

We note that $\sigma_t = \phi_{t\#}\sigma_a$ solves the continuity equation (4.1.8)(see [46], Page 167).

A simple computation gives

$$\begin{aligned} \text{div } [\mathbf{v}_t] = \frac{1}{\sqrt{\Upsilon}} F + \frac{r_0^3 \sqrt{\Upsilon}}{\left(1 - 2r_0^2 \frac{\partial \Psi}{\partial \Upsilon}\right)^{\frac{3}{2}}} \frac{\partial^2 \Psi}{\partial \Upsilon^2} \frac{\partial F}{\partial r} + \frac{\partial^2 \Psi}{\partial Z^2} \frac{\partial S}{\partial z} + \\ \frac{\partial^2 \Psi}{\partial \Upsilon \partial Z} \left[2\sqrt{\Upsilon} \frac{\partial F}{\partial z} + \frac{\partial S}{\partial r} \frac{r_0^3}{\left(1 - 2r_0^2 \frac{\partial \Psi}{\partial \Upsilon}\right)^{\frac{3}{2}}} \right] \end{aligned} \tag{4.1.10}$$

As $(A2)$ holds

$$\text{div }[\mathbf{v}_t] = \frac{1}{\sqrt{\Upsilon}}F + \frac{r_0^3\sqrt{\Upsilon}}{\left(\sqrt{1-2r_0^2\frac{\partial\Psi}{\partial\Upsilon}}\right)^3}\frac{\partial^2\Psi}{\partial\Upsilon^2}\frac{\partial F}{\partial r} + \frac{\partial^2\Psi}{\partial Z^2}\frac{\partial S}{\partial z}$$

Since Ψ_t is convex, its second partial derivatives with respect to Υ and Z are all non negative. This, combined with $(A3)$ leads to

$$\text{div }[\mathbf{v}_t] \geq 0$$

It is well known that

$$\det \nabla\phi_t = \det \nabla\phi_a \exp\left(\int_a^t \text{div }[\mathbf{v}_s] \circ \phi_s ds\right)$$

Therefore, as div $[\mathbf{v}_t] \geq 0$, $t \longmapsto \det(\nabla\phi_t)$ is non decreasing and so

$$\det(\nabla\phi_t) \geq \det(\nabla\phi_a) = 1 \tag{4.1.11}$$

2. We use $(A1)$ and the definition of the flow in (4.1.9) to obtain that

$$0 \leq \dot{\phi}_{1t} \leq 2\sqrt{\phi_{1t}}M, \qquad 0 \leq \dot{\phi}_{2t} \leq M$$

or

$$0 \leq \frac{d}{dt}\sqrt{\phi_{1t}} = \frac{\dot{\phi}_{1t}}{2\sqrt{\phi_{1t}}} \leq M \qquad 0 \leq \dot{\phi}_{2t} \leq M$$

By integrating this equation from a to t, for each $(\Upsilon, Z) \in \mathbb{R}_+^{*2}$, we obtain

$$0 \leq \sqrt{\phi_{1t}}(\Upsilon, Z) - \sqrt{\Upsilon} \leq M(t-a), \qquad Z \leq \phi_{2t}(\Upsilon, Z) \leq Z + M(t-a)$$

that is,

$$\Upsilon \leq \phi_{1t}(\Upsilon, Z) \leq (\sqrt{\Upsilon} + M(t-a))^2, \qquad Z \leq \phi_{2t}(\Upsilon, Z) \leq Z + M(t-a) \tag{4.1.12}$$

And so, for $(\Upsilon, Z) \in [0, L_a]^2$, (4.1.12) implies that

$$0 \leq \phi_{1t}(\Upsilon, Z) \leq (\sqrt{L_a} + M(t-a))^2, \qquad 0 \leq \phi_{2t}(\Upsilon, Z) \leq L_a + M(t-a) \tag{4.1.13}$$

Since $L_a > 1$, (4.1.13) implies that

$$\phi_t([0, L_a]^2) \subset [0, L_a(1 + M(t - a))^2]^2$$

Therefore, as $\sigma_t = \phi_{t\#}\sigma_a$ and ϕ_t is continuous,

$$spt(\sigma_t) = \overline{\phi_t\left(spt(\sigma_a)\right)} \subset \phi_t([0, L_a]^2) \subset [0, L_a(1 + M(t - a))^2]^2$$

3. In view of (4.1.11), $\sigma_t = \phi_{t\#}\sigma_a$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^2$ and its density function ϱ_t satisfies

$$\varrho_t \circ \phi_t = \frac{\varrho_a}{\det(\nabla\phi_t)} \leq \varrho_a \tag{4.1.14}$$

Therefore, using the equality in (4.1.14) we obtain

$$\varrho_t^r = \varrho_a^r \circ \phi^{-1}\left(\det[\nabla\phi_t]^{-1}\right)^r \circ \phi^{-1} = \varrho_a^r \circ \phi^{-1}\left(\det[\nabla\phi_t]^{-1}\right) \circ \phi^{-1}\left(\det[\nabla\phi_t]^{-1}\right)^{r-1} \circ \phi^{-1} \tag{4.1.15}$$

As $\det[\nabla\phi_t] \geq 1$, (4.1.15) implies that

$$\varrho_t^r \leq \varrho_a^r \circ \phi^{-1} \det[\nabla\phi]^{-1} \circ \phi^{-1} \tag{4.1.16}$$

for any $r \geq 1$. We exploit (4.1.16) to obtain that

$$\int_{\mathbb{R}^2} \varrho_t^r dq \leq \int_{\mathbb{R}^2} \varrho_a^r \circ \phi^{-1} \det[\nabla\phi]^{-1} \circ \phi^{-1} dq = \int_{\mathbb{R}^2} \varrho_a^r dq$$

This establishes (a). We easily check

$$|\mathbf{v}| \leq M\sqrt{4L_a + 1} = c_0$$

and so, by (4.1.6)

$$W_1(\sigma_t, \sigma_s) \leq \int_s^t ||\mathbf{v}_r||_{L^1(\sigma_r)} dr \leq c_0(t - s) \tag{4.1.17}$$

for all $a \le s \le t \le a + \tau$.

Therefore $t \longrightarrow \sigma_t$ is c_0-Lipschitz continuous on $[a, a+\tau]$. Thus ,

$$W_1(\bar{\sigma}_a, \sigma_t) \le c_0(t-a) \le c_0\tau \tag{4.1.18}$$

for all $t \in [a, a+\tau]$. As a consequence $\{\sigma_t\}_{t\in[a,a+\tau]}$ is bounded in the $1-$Wasserstein space.

Step2 We consider now the general case where Ψ is not necessary smooth. We note that, as $\Psi(t,\cdot)$ is convex, $\Psi(t,\cdot)$ is locally lipschitz and so $\Psi(t,\cdot) \in W^{1,1}_{loc}(\mathbb{R}^{*2}_+)$ for each $t \ge 0$ fixed. We set

$$\Psi^n(t,\cdot) := \Psi(t,\cdot) * j_n$$

Here, $\{j_n\}_{n=1}^{\infty}$ are the standard mollifiers. We obtain that $\Psi_n(t,\cdot)$ converges to $\Psi(t,\cdot)$ in $W^{1,1}_{loc}(\mathbb{R}^{*2}_+)$. This convergence guarantees that up to a subsequence $\nabla\Psi_n(t,\cdot)$ converges $\nabla\Psi(t,\cdot)$, $a.e$ in $\mathbb{R}^{*2}_+$.

Let's denote by $\mathbf{v}_n$ the velocity field when Ψ is replaced by Ψ_n in (4.1.7). Without loss of generality, we have that

$$\mathbf{v}_n \longrightarrow \mathbf{v} \qquad a.e$$

Let $\sigma^n = \varrho^n\mathcal{L}^2$ denotes the solution of (4.1.8) when $\mathbf{v}$ is replaced by $\mathbf{v}_n$. Then σ^n satisfies (4.1.17) and the conditions (a), (b) and (c). We obtain that the family $t \longrightarrow \sigma^n_t$ is equi-Lipschitz on $[a, a+\tau]$ with respect to W_1 and (4.1.18) ensures that it is equibounded in $\mathcal{P}(\mathbb{R}^2)$ with respect to W_1. Therefore, there exists a subsequence that we still denote by $t \longrightarrow \sigma^n_t$ (n is independent of t) such that $\{\sigma^n_t\}_{n=1}^{\infty}$ converge narrowly to σ_t for each $t \in [0, \tau]$.

Since the Wasserstein distance is lower semi-continuous with respect to narrow convergence and σ^n_t satisfy (4.1.17), σ_t also satisfies (4.1.17), that is, σ_t is c_0-lipschitz continuous on $(a, a+\tau)$. By condition (a), $\{\varrho^n_t\}_{n=1}^{\infty}$ is equibounded in L^r, $r \ge 1$ and so, as $\{\varrho^n_t\}_{n=1}^{\infty}$ converges weakly* to σ_t, the Dunford-Pettis theorem guarantees that

σ_t is absolutely continuous with respect to Lebesgue , that is $\sigma_t = \varrho_t \mathcal{L}^2$. Also, as $\{\varrho_t^n\}_{n=1}^{\infty}$ satisfy the condition (a), the weakly lower semi-contnuity of the L^r norms ensures that ϱ_t satisfy the condition (a) as well.

To obtain the continuity equation in (c), we only need to show that $\{\mathbf{v}_t^n \sigma_t^n\}_{n=1}^{\infty}$ converges to $\mathbf{v}_t \sigma_t$ in the sense of distribution for each t fixed, as $\{\mathbf{v}_t^n \sigma_t^n dt\}_{n=1}^{\infty}$ converges to $\mathbf{v}_t \sigma_t dt$ in the sense of distribution will be obtained by a simple application of Lebesgue dominated convergnce. We note that the inequality in (a) ensures that $\{\varrho_t^n\}_{n=1}^{\infty}$ is equi- integrable. As $\mathbf{v}_t^n$ converges to $\mathbf{v}_t$ and $\sigma^n = \varrho^n \mathcal{L}^2$ narrowly to $\sigma = \varrho \mathcal{L}^2$ we use the lemma 4.1.1 to obtain the desired result □

Let $\sigma \in \Sigma$. Set $(P, \Psi) = \bar{\mathcal{H}}(\sigma)$ and define

$$\mathbf{X}_t[\sigma] := \left(2\sqrt{\Upsilon} \left[\bar{F}_t \left(\frac{r_0}{\sqrt{1 - 2r_0^2 \frac{\partial \Psi}{\partial \Upsilon}}}, \frac{\partial \Psi}{\partial Z} \right) \right], \; \frac{g}{\theta_0} \bar{S}_t \left(\frac{r_0}{\sqrt{1 - 2r_0^2 \frac{\partial \Psi}{\partial \Upsilon}}}, \frac{\partial \Psi}{\partial Z} \right) \right) \tag{4.1.19}$$

Theorem 4.1.5 *Assume that (FS1),(FS2) and (FS3) hold. Assume* $0 < L_0 < R_0$ *and let* $\bar{\sigma}_0 = \bar{\varrho}_0 \mathcal{L}^2 \in \Sigma$ *such that*

$$\operatorname{spt}(\bar{\sigma}_0) \subset [0, L_0]^2$$

Let $T > 0$ *such that* $L_0 e^{6MT} \leq R_0$ *. Then, there exists* $\sigma_t = \varrho_t \mathcal{L}^2 \in \Sigma$ *satisfying :*

(a) $\int_{\mathbb{R}^2} \varrho_t^r dq \leq \int_{\mathbb{R}^2} \bar{\varrho}_0^r dq$ *for any* $r \geq 1$

(b) $t \longmapsto \sigma_t \in AC_2\left((0,T); \mathcal{P}(\mathbb{R}_+^2)\right)$ *and*

$$\begin{cases} \frac{\partial \sigma}{\partial t} + \operatorname{div}(\sigma \mathbf{X}_t[\sigma]) = 0, & (0,T) \times \mathbb{R}^2 \\ \sigma_{|t=0} = \bar{\sigma}_0 \end{cases} \tag{4.1.20}$$

holds in the sense of distribution.

(c) $t \longmapsto \sigma_t$ *is lipschitz continuous with lipschitz constant less than* c_0.

Proof: We fix a non negative integer N. and divide the interval $[0, T]$ into N intervals with equal lenght $\tau = \frac{T}{N}$. We first show that we can construct a discrete function

$\sigma_t^N = \varrho_t^N \mathcal{L}^2$ satisfying the following proprieties:

(a1) $\int_{\mathbb{R}^2} (\varrho_t^N)^r dq \leq \int_{\mathbb{R}^2} (\varrho_0)^r dq$ for any $r \geq 1$

(b1) The "delayed" equation

$$\begin{cases} \frac{\partial \sigma_t^N}{\partial t} + \operatorname{div}(\sigma_t^N \mathbf{v}_t^N) = 0, & (0,T) \times \mathbb{R}^2 \\ \\ \sigma_{|t=0} = \bar{\sigma}_0 \end{cases} \tag{4.1.21}$$

holds in the sense of distribution and $\mathbf{v}_t^N = \mathbf{X}_{[\frac{t}{\tau}]\tau}[\sigma^N_{[\frac{t}{\tau}]\tau}]$ for all $t \in [0,T)$.

(c1) $t \longmapsto \sigma_t^N$ is lipschitz continuous with respect to W_1 with lipschitz constant less than c_0.

The construction of σ_t^N goes as follows: we start off by setting $\sigma_0^N = \bar{\sigma}_0$ and $\mathbf{v}_t^N = \mathbf{X}_0[\bar{\sigma}_0]$ for $t \in [0,\tau]$. we use lemma 4.1.3 to obtain a solution σ_t^N on $[0,\tau]$. We repeat inductively the same process $(N-1)$ times by setting $\sigma_{i\tau}^N = \sigma_{i\tau}$ and $\mathbf{v}_t^N = \mathbf{X}_{i\tau}[\sigma_{i\tau}]$ for $t \in [i\tau, (i-1)\tau]$ and using lemma 4.1.3 to obtain σ_t^N on $t \in [i\tau, (i+1)\tau]$. In view of lemma 4.1.3, we note that the process described above works as long as $\{\sigma_{i\tau}\}_{1 \leq i \leq N}$ stays absolutely continuous with respect to Lebesgue and compactly supported in $\mathbb{R}^2_+$. We next show that $\{\sigma_{i\tau}\}_{1 \leq i \leq N} \subset \Sigma$. We first observe that by construction, lemma 4.1.3 guarantees that $\{\sigma_{i\tau}\}_{1 \leq i \leq N}$ are absolutely continuous with respect to Lebesgue in $\mathbb{R}^2_+$

Define

$$L_i := \max\left(\sup\{\Upsilon : (\Upsilon, Z) \in \operatorname{spt}(\sigma_{i\tau})\}; \sup\{Z : (\Upsilon, Z) \in \operatorname{spt}(\sigma_{i\tau})\}\right)$$

for $1 \leq i \leq N$. By lemma 4.1.3,

$$L_{i+1} \leq L_i(1+M\tau)^2 \leq L_0(1+M\tau)^{2(i+2)} < L_0(1+M\tau)^{6N} = L_0(1+M\frac{T}{N})^{6N} \leq L_0 e^{6MT}.$$

With the constraint $L_0 e^{6MT} < R_0$, on T, we obtain that for any $0 \leq i \leq N$, $\operatorname{spt}(\sigma_{i\tau})$ is contained in $[0,R_0]^2$. Therefore the above construction of σ_t^N is thoroughly justified. We easily check that the conditions (a1) and (c1) follow from the condition (a) and (c) of lemma 4.1.3

Step 2 By (c1), $t \longmapsto \sigma_t^N$ are equi-Lipschitz continuous on $[0,T]$ and since $\sigma_0^N = \bar{\sigma}_0$ for all N they are equibounded in the 1-Wasserstein space. Thus, there exists a subsequence of $t \longmapsto \sigma_t^N$ still denoted by $t \longmapsto \sigma_t^N$ (N independent of t) such that $\left\{\sigma_t^N\right\}_{N=1}^{\infty}$ convereges narrowly to σ for any $t \in [0,T]$.

In view of (a1), the theorem of Dunford-Pettis ensures that $\sigma_t = \rho_t \mathcal{L}^2$. The lower semi -continuity of the L^r- norms leads to (a). We next show that σ_t satisfies (4.1.20). It suffices to show that there exists a subsequence $\{\sigma_{N_i}\}_{i=1}^{\infty}$ of $\{\sigma_N\}_{N=1}^{\infty}$ such that $\mathbf{v}^{N_i}\sigma_t^{N_i}$ converges to $\mathbf{v}_t\sigma_t$ in the sense of distribution. As $\{\sigma_{N_i}\}_{i=1}^{\infty}$ converges to σ , by lemma 3.4.3, $\nabla\Psi_{N_i}$ converges to $\nabla\Psi$ so that $\left\{\mathbf{v}^{N_i}\right\}_{i=1}^{\infty}$ converges $\mathbf{X}_t[\sigma]$ $\mathcal{L}^2 - a.e.$ We replace N by N_i in (4.1.21) and let $N_i \to \infty$ to obtain (4.1.20).

4.2 Existence of a solution for general initial data.

In this section, we impose the following conditions on the forcing terms $\bar{F}$ and $\bar{S}$: $\mathbb{R}_+ \times \mathbb{R}^2 \longrightarrow \mathbb{R}$.

- (B_1) $\bar{F}$ and $\bar{S}$ are continuous and bounded.
- (B_2) $\bar{F} \geq 0$ and $\bar{S} \geq 0$

Let $\sigma \in \mathcal{P}([0,R_0]^2)$. If $\bar{\mathcal{H}}(\sigma) = (P,\Psi)$ then $(P,\Psi) \in \mathcal{U}_0$ and ∇P has values in $[0,R_0]^2$.

Recall

$$\mathbf{d}(s,z) = \left(\frac{r_0}{\sqrt{1-2r_0^2 s}}, z\right)$$

Set

$$\mathbb{F}_t = (F_t \circ \mathbf{d}, S_t \circ \mathbf{d})$$

As $\bar{F}$ and $\bar{S}$ are bounded, there exists a constant C_0 (independent of t) such that

$$||\mathbb{F}_t|||_\infty \leq \frac{C_0}{\sqrt{R_0}}$$

for all $t \geq 0$. To any function $G = (G_1, G_2)$ we associate $\mathcal{A}[G]$ defined by

$$\mathcal{A}[G](\Upsilon, Z) = \left(\sqrt{\Upsilon} G_1(\Upsilon, Z), G_2(\Upsilon, Z)\right).$$

Note that if $G \in C([0, R_0]^2)$ then

$$\mathcal{A}[G] \in C([0, R_0]^2) \qquad \text{with} \qquad \|\mathcal{A}[G]\|_\infty \leq \sqrt{R_0}\|G\|_\infty$$

If $h = \mathcal{H}(\sigma)$ and $(P, \Psi) = \bar{\mathcal{H}}(\sigma)$ then we define

$$L_t[\sigma](G) = \int_{\mathbb{R}^2} \langle \mathcal{A}[G] \circ \nabla P,\ \mathbb{F}_t \rangle e(s) \chi_{D_h}(s, z) ds dz$$

for all $G \in L^1(\sigma, \mathbb{R}^2)$. Note that if $G \in L^1(\sigma, \mathbb{R}^2)$ then

$$\mathcal{A}[G] \in L^1(\sigma; \mathbb{R}^2) \qquad \text{with} \qquad \|\mathcal{A}[G]\|_{L^1(\sigma;\mathbb{R}^2)} \leq \sqrt{R_0}\|G\|_{L^1(\sigma;\mathbb{R}^2)}$$

for all $G \in L^1(\sigma, \mathbb{R}^2)$. Observe that if $G \in L^1(\sigma, \mathbb{R}^2)$ such that $G_1 \geq 0, G_2 \geq 0$ then $L_t[\sigma](G) \geq 0$.

Lemma 4.2.1 *Fix $t > 0$. Let $\sigma \in \mathcal{P}([0, R_0]^2)$.*
Then, there exists $V_t[\sigma] = (V_t^1[\sigma], V_t^2[\sigma]) \in L^\infty(\sigma; \mathbb{R}^2)$ such that

$$L_t[\sigma](G) = \int_{\mathbb{R}^2} \langle V_t[\sigma], G \rangle d\sigma \tag{4.2.1}$$

for all $G \in L^1(\sigma, \mathbb{R}^2)$.

$$||V_t[\sigma]||_{L^\infty(\sigma;\mathbb{R}^2)} \leq C_0 \tag{4.2.2}$$

and

$$V_t^1[\sigma] \geq 0 \qquad V_t^2[\sigma] \geq 0 \quad \sigma\ a.e \tag{4.2.3}$$

Proof:

$$
\begin{aligned}
|L_t[\sigma](G)| &\leq \|\mathbb{F}_t\|_\infty \int_{\mathbb{R}^2} |\mathcal{A}[G]| \circ \nabla P \ \ e(s)\chi_{D_h} dsdz \\
&= \|\mathbb{F}_t\|_\infty \int_{\mathbb{R}^2} |\mathcal{A}[G]| \, d\sigma \\
&= \|\mathbb{F}_t\|_\infty \sqrt{R_0} \|G\|_{L^1(\sigma;\mathbb{R}^2)} \\
&\leq C_0 \|G\|_{L^1(\sigma;\mathbb{R}^2)}
\end{aligned}
$$

By the Riesz representation theorem for linear functionals, we obtain that there exists $V_t[\sigma]$ such that (4.2.1) and (4.2.2) holds. Note that

$$
L_t[\sigma](G_1, 0) = \int_{\mathbb{R}^2} \langle V_t[\sigma], (G_1, 0)\rangle d\sigma = \int_{V_t^1[\sigma] \geq 0} V_t^1[\sigma] G_1 d\sigma + \int_{V_t^1[\sigma] < 0} V_t^1[\sigma] G_1 d\sigma
$$

Choose $G_1 = \chi_{\{V_t^1[\sigma] < 0\}} \geq 0$ so that $L_t[\sigma](G_1, 0) \geq 0$. If $\sigma(V_t^1[\sigma] < 0) > 0$ then

$$
0 \leq L_t[\sigma](G_1, 0) = \int_{\{V_t^1[\sigma] < 0\}} V_t^1[\sigma] d\sigma < 0
$$

Therefore, $V_t^1[\sigma] \geq 0$ σ a.e. A similar argument shows that $V_t^2[\sigma] \geq 0$ σ a.e. $\square$

Remark 4.2.2 *Let $\sigma \in \mathcal{P}([0, R_0]^2)$ and $h = \mathcal{H}(\sigma)$ and $(P, \Psi) = \bar{\mathcal{H}}(\sigma)$. Then for $t, r \geq 0$*

$$
L_t[\sigma](G) - L_r[\sigma](G) = \int_{\mathbb{R}^2} \langle \mathcal{A}[G] \circ \nabla P, \ \mathbb{F}_t - \mathbb{F}_r \rangle e(s) \chi_{D_h}(s, z) dsdz
$$

and so in view of lemma 4.2.1,

$$
\begin{aligned}
\left| \int_{\mathbb{R}^2} \langle V_t[\sigma] - V_r[\sigma], G \rangle d\sigma \right| &= |L_t[\sigma](G) - L_r[\sigma](G)| \\
&\leq \|\mathcal{A}[G]\|_\infty \sup_{p \in \Delta_{r_0}} |\mathbb{F}_t(p) - \mathbb{F}_r(p)| \int_{\mathbb{R}^2} e(s) \chi_{D_h}(s, z) dsdz \\
&\leq \|G\|_\infty \sup_{p \in \Delta_{r_0}} |\mathbb{F}_t(p) - \mathbb{F}_r(p)|
\end{aligned}
\tag{4.2.4}
$$

Lemma 4.2.3 *Let $t \geq 0$. Let $\{\sigma_n\}_{n=1}^\infty$ and σ be elements of $\mathcal{P}([0, R_0]^2)$ such that $\{\sigma_n\}_{n=1}^\infty$ converges narrowly to σ. Then $\{V_t[\sigma_n]\sigma_n\}_n$ converges to $V_t[\sigma]\sigma$ in the sens of distributions.*

Proof: Let $(P_n, \Psi_n) = \mathcal{H}(\sigma_n)$, $(P, \Psi) = \mathcal{H}(\sigma)$. We extract from $\{\sigma_n\}_{n=1}^{\infty}$ a subsequence that we still denote by $\{\sigma_n\}_{n=1}^{\infty}$. As $\{\sigma_n\}_{n=1}^{\infty}$ converges narrowly to σ, lemma 3.4.2 ensures that there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of integers such that $\{P_{n_k}\}_{k=1}^{\infty}$ converges uniformly. Hence, by lemma 3.4.3 $0 \leq h, h_{n_k} \leq M_0 < \frac{1}{2r_0^2}$ for some constant M_0 and so $\{e(s)\chi_{D_h}\}_{k=1}^{\infty}$ is equi-integrable. Lemma 3.4.3 ensures that $\{\nabla P_{n_k}\}_{k=1}^{\infty}$ converges a.e to ∇P. Let $G \in C([0, R_0]^2)$. Then $\mathcal{A}[G]$ is continuous on $[0, R_0]^2$ and $\langle \mathcal{A}[G] \circ \nabla P_{n_k}; \mathbb{F}_t\rangle$ converges a.e to $\langle \mathcal{A}[G] \circ \nabla P; \mathbb{F}_t\rangle$. Moreover, as G is bounded function, $\mathcal{A}[G]$ is bounded. In addition, since $\mathbb{F}$ is bounded, there exists $M > 0$ such that $|\langle \mathcal{A}[G] \circ \nabla P_{n_k}; \mathbb{F}_t\rangle| \leq M$ for all $k \geq 1$ and $t > 0$. Using lemma 4.1.1 , we obtain that

$$\lim_{k\to\infty} \int \langle \mathcal{A}[G] \circ \nabla P_{n_k}; \mathbb{F}\rangle e(s)\chi_{D_{h_{n_k}}}(s, z)dsdz = \int \langle \mathcal{A}[G] \circ \nabla P; \mathbb{F}\rangle e(s)\chi_{D_h}(s, z)dsdz$$

This, in light of (4.2.1) becomes

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} \langle V_t[\sigma_{n_k}], G\rangle d\sigma_{n_k} = \int_{\mathbb{R}^2} \langle V_t[\sigma], G\rangle d\sigma$$

As G is arbitrary, we obtain that $\{V_t[\sigma_{n_k}]\sigma_{n_k}\}_k$ converges to $V_t[\sigma]\sigma$ in the sens of distribution. Since the limit $V_t[\sigma]\sigma$ is independent of the extracted subsequence of $\{V_t[\sigma_n]\sigma_n\}_n$, we conclude that the whole sequence $\{V_t[\sigma_n]\sigma_n\}_n$ converges narrowly to $V_t[\sigma]\sigma$.

Definition 4.2.4 *Let $T > 0$. $t \longrightarrow \sigma_t$ is an absolutely continuous path in $\mathcal{P}([0, R_0]^2)$.*

Let $(P(t, \cdot), \Psi(t, \cdot)) = \bar{\mathcal{H}}(\sigma_t)$ and $h_t = \mathcal{H}(\sigma_t)$ We say that

$$\dot{\sigma_t} = \chi_{\mathcal{H}}^{\bar{\mathcal{H}}}(\sigma_t)$$

in the weak dual sense if

$$\int_0^T dt \int_{\mathbb{R}^2} \frac{\partial \varphi}{\partial t} \circ \nabla P(t, \cdot)e(s)\chi_{D_{h_t}} dsdz + \int_0^T L_t[\sigma](\nabla\varphi)dt = 0 \qquad (4.2.5)$$

for all $\varphi \in C^1((0, T) \times \mathbb{R}^2)$.

Remark 4.2.5 *This definition is natural in the sense that if σ_t is absolutely continuous with respect to lebesgue then we recover the continuity equation in (4.1.20). Indeed, if σ_t is absolutely continuous with respect to lebesgue, then $\nabla\Psi_t \circ \nabla P_t = \mathbf{id}$ $e(s)\chi_{D_{h_t}}$ a.e (see Proposition 3.3.17 (ii)). Thus,*

$$L_t[\sigma](\nabla\varphi_t) = \int_{\mathbb{R}^2} \langle \mathcal{A}[\nabla\varphi_t] \circ \nabla P_t,\ \mathbb{F}_t\rangle e(s)\chi_{D_{h_t}} dsdz = \int_{\mathbb{R}^2} \langle \mathcal{A}[\nabla\varphi_t],\ \mathbb{F}_t \circ \nabla\Psi_t\rangle d\sigma$$

for all $\varphi \in C^1((0,T) \times \mathbb{R}^2)$. And so, (4.2.5) becomes

$$\int_0^T dt \int_{\mathbb{R}^2} \frac{\partial\varphi}{\partial t} + \sqrt{\Upsilon} F_t \circ \mathbf{d}_1 \circ \nabla\Psi(\Upsilon, Z)\frac{\partial\varphi}{\partial\Upsilon} + \frac{g}{\theta_0} S_t \circ \mathbf{d}_1 \circ \nabla\Psi \frac{\partial\varphi}{\partial Z} d\sigma_t dt = 0$$

for all $\varphi \in C^1((0,T) \times \mathbb{R}^2)$. That is,

$$\frac{\partial\sigma}{\partial t} + div(\sigma\mathbf{X}_t[\sigma]) = 0, \qquad (0,T) \times \mathbb{R}^2$$

holds in the distribution sense when $\mathbf{X}_t[\sigma]$ is given by (4.1.19)

Remark 4.2.6 *In view of lemma 4.2.1 , (4.2.5) becomes*

$$\int_0^T dt \int_{\mathbb{R}^2} \frac{\partial\varphi}{\partial t} + \langle \nabla\varphi, V_t[\sigma]\rangle d\sigma_t dt = 0$$

for all $\varphi \in C^1((0,T) \times \mathbb{R}^2)$. That is,

$$\frac{\partial\sigma}{\partial t} + div(\sigma V_t[\sigma]) = 0, \qquad (0,T) \times \mathbb{R}^2 \tag{4.2.6}$$

holds in the distribution sense.

Lemma 4.2.7 *let be f a borel map, $\mu \in \mathcal{P}(\mathbb{R}^2)$, and $\mathbf{v} \in L^\infty(\mu, \mathbb{R}^2; \mathbb{R}^2)$. Then setting $\nu = f_\#\mu$, we have $f_\#(\mathbf{v}\mu) = \mathbf{w}\nu$ for some $\mathbf{w} \in L^\infty(\nu, \mathbb{R}^2; \mathbb{R}^2)$ with*

$$||\mathbf{w}||_{L^\infty(\nu,\, \mathbb{R}^2;\mathbb{R}^2)} \leq ||\mathbf{v}||_{L^\infty(\mu,\, \mathbb{R}^2;\mathbb{R}^2)}$$

Proof: Let $\varphi \in L^1(\nu, \mathbb{R}^2; \mathbb{R}^2)$. We have

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \varphi d(f_{\#}\mathbf{v}\mu)\right| &= \left|\int_{\mathbb{R}^2} \langle \varphi \circ f, \mathbf{v}\rangle d\mu\right| \\ &\leq ||\mathbf{v}||_{L^\infty(\mu,\, \mathbb{R}^2;\mathbb{R}^2)} \int_{\mathbb{R}^2} |\varphi \circ f| d\mu \\ &\leq ||\mathbf{v}||_{L^\infty(\mu,\, \mathbb{R}^2;\mathbb{R}^2)} \int_{\mathbb{R}^2} |\varphi| d\nu\end{aligned} \tag{4.2.7}$$

By Riesz representation theorem for linear functionals, there exists $\mathbf{w} \in L^\infty(\nu, \mathbb{R}^2; \mathbb{R}^2)$ such that

$$\int \varphi d(f_{\#}\mathbf{v}\mu) = \int_{\mathbb{R}^2} \langle \varphi, \mathbf{w}\rangle d\nu$$

and

$$||\mathbf{w}||_{L^\infty(\nu,\, \mathbb{R}^2;\mathbb{R}^2)} \leq ||\mathbf{v}||_{L^\infty(\mu,\, \mathbb{R}^2;\mathbb{R}^2)}$$

□

Theorem 4.2.8 *Assume $\bar{F}$ and $\bar{S}$ satisfy (B_1) and (B_2). Assume $0 < L_0 < R_0$. Let $\bar{\sigma}_0 \in \mathcal{P}(\mathbb{R}^2_+)$ such that $\mathrm{spt}(\bar{\sigma}_0) \subset [0, L_0]^2$. Let $T > 0$ such that $L_0 + C_0 T < R_0$, Then there exists $\sigma_t : [0, T] \longmapsto \big(\mathcal{P}(\mathbb{R}^2_+), W_1\big)$ C_0-lipschitz continuous such that $\mathrm{spt}(\sigma_t) \subset [0, R_0]^2$ and*

$$\begin{cases} \dot{\sigma}_t = \chi_{\mathcal{H}}^{\bar{\mathcal{H}}}(\sigma_t) \\ \sigma_{|t=0} = \bar{\sigma}_0 \end{cases} \tag{4.2.8}$$

Proof: Step1 (Construction of a discrete solution) We choose $V = (V^1, V^2)$ as provided by lemma 4.2.1. For any $\sigma \in \mathcal{P}([0, R]^2)$, by redefining $V_t[\sigma]$ on a σ negligible subset of $\mathbb{R}^2$, we may assume without loss of generality that $V_t^1[\sigma], V_t^2[\sigma] \geq 0$ and $|V_t[\sigma]| \leq C_0$ on $\mathbb{R}^2$ all $t \geq 0$. Let N be a positive integer. We first build a solution σ_t^N satisfying

(a) $t \longmapsto \sigma_t^N$ is Lipschitz continuous with the Lipschitz constant less than or equal to C_0

(b) $spt(\sigma_t^N) \subset [0, R_0]^2$ for $t \in [0, T]$

(c) $t \longmapsto \sigma_t^N$ satisfies

$$\begin{cases} \frac{\partial \sigma^N}{\partial t} + \mathrm{div}(\sigma^N \mathbf{w}_t^N) = 0, & (0,T) \times \mathbb{R}^2 \\ \sigma^N_{|t=0} = \bar{\sigma}_0 \end{cases}$$

holds in the distributional sense with

$$\mathbf{w}_t^N \sigma_t^N = \left(\mathbf{id} + (t - [\frac{t}{\tau}]\tau) V_{[\frac{t}{\tau}]\tau}[\sigma^N_{[\frac{t}{\tau}]\tau}]\right)_{\#} \left(V_{[\frac{t}{\tau}]\tau}[\sigma^N_{[\frac{t}{\tau}]\tau}]\sigma^N_{[\frac{t}{\tau}]\tau}\right) \tag{4.2.9}$$

We construct the solution on $[0, \tau]$. We set $\mathbf{w}_0^N = V_0[\bar{\sigma}_0]$ and consider the following

$$\sigma_t^N = \left(\mathbf{id} + t\mathbf{w}_0^N\right)_{\#} \bar{\sigma}_0 \qquad \mathbf{w}_t^N = \frac{\left(\mathbf{id} + t\mathbf{w}_0^N\right)_{\#} (\mathbf{w}_0^N \bar{\sigma}_0)}{\sigma_t^N} \qquad t \in [0, \tau]$$

Clearly, σ_t^N and $\mathbf{w}_t^N$ above solve the equation in (c). To see this, we choose $\varphi \in C_c^\infty(\mathbb{R}^2)$ and note that

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^2} \varphi d\sigma_t^N = & \quad \frac{d}{dt}\int_{\mathbb{R}^2} \varphi(\mathbf{id} + t\mathbf{w}_0^N) d\bar{\sigma} = \int_{\mathbb{R}^2} < \nabla\varphi(x + t\mathbf{w}_0^N); \mathbf{w}_0^N > d\bar{\sigma} \\ = & \quad \sum_{k=1}^{2} \int_{\mathbb{R}^2} \frac{\partial \varphi}{\partial x_i} d\left(\left(\mathbf{id} + t\mathbf{w}_0^N\right)_{\#} (\mathbf{w}_0^N \bar{\sigma}_0)\right) = \int_{\mathbb{R}^2} < \nabla\varphi; \mathbf{w}_\tau^N > d\sigma_t^N \end{aligned} \tag{4.2.10}$$

As φ is arbitrary, we obtain the continuity equation in (a) on the interval $[0, \tau]$. Note that the components of $\mathbf{w}_0^N$ on $\mathbb{R}^2$ are all non negative and bounded by C_0. As $\mathrm{spt}(\bar{\sigma}_0) \subset [0, L_0]^2$ and $\sigma_t^N = \left(\mathbf{id} + t\mathbf{w}_0^N\right)_{\#} \bar{\sigma}_0$, $\mathrm{spt}(\sigma_t^N) \subset [0, L_0 + C_0\tau]^2$ for all $t \in [0, \tau]$. Moreover,

$$||\mathbf{w}_t^N||_{L^\infty(\sigma_t, \mathbb{R}^2)} \leq ||\mathbf{w}_0^N||_{L^\infty(\sigma_0, \mathbb{R}^2)} = ||V_0[\sigma_0]||_{L^\infty(\sigma_0, \mathbb{R}^2)} \leq C_0 \tag{4.2.11}$$

for $t \in [0, \tau]$. The first inequality in (4.2.11) is ensured by lemma 4.2.7 and the second inequality comes from (4.2.2).

As σ_t^N and $\mathbf{w}_t^N$ solve the continuity equation in (c) on $(0, \tau)$, we use (4.2.11) and (4.1.6) to obtain

$$W_1^(\sigma_t^N, \sigma_s^N) \leq \int_s^t ||\mathbf{w}_r^N||_{L^1(\sigma_t,\mathbb{R}^2)} dr \leq C_0(s-t) \qquad 0 \leq t \leq s \leq \tau.$$

Therefore $t \longmapsto \sigma_t^N$ is lipschitz continuous on $[0,\tau]$. In particular, $W(\bar{\sigma}, \sigma_t^N) \leq C_0\tau$ for all $t \in [0,\tau]$.

We can repeat this process by setting $\mathbf{w}_\tau^N = V_\tau[\sigma_\tau^N]$ and by extending the solution to $[\tau, 2\tau]$ in the following way:

$$\sigma_t^N = \big(\mathbf{id} + (t-\tau)\mathbf{w}_\tau^N\big)_{\#}\sigma_\tau \qquad\qquad \mathbf{w}_t^N = \frac{\big(\mathbf{id} + (t-\tau)\mathbf{w}_\tau^N\big)_{\#}(\mathbf{w}_\tau^N\bar{\sigma}_\tau)}{\sigma_t^N} \qquad t \in [\tau, 2\tau]$$

A computation analogous to the one in (4.2.10) shows that this extension solves the continuity equation in (c) on $(\tau, 2\tau)$. We also have $spt(\sigma_t^N) \subset [0, L_0 + 2C_0\tau]^2$ for all $t \in [\tau, 2\tau]$ and

$$W_1(\sigma_t^N, \sigma_s^N) \leq C_0(s-t) \qquad \tau \leq t \leq s \leq 2\tau.$$

By iterating this process $N-2$ more times by setting $\mathbf{w}_{k\tau}^N = V_{k\tau}[\sigma_{k\tau}]$, $k = 3, ..N$ we build a solution $t \longmapsto \sigma_t^N$ to the continuity equation on $[0,T]$. Futhermore, this solution satisfies the following

$$W_2(\sigma_t^N, \sigma_0) \leq C_0 T, \qquad ||\mathbf{w}_t^N||_{L^2(\sigma_t,\mathbb{R}^2)} \leq C_0 \tag{4.2.12}$$

$$\mathrm{spt}(\sigma_t^N) \subset [0, L_0 + NC_0\tau]^2 = [0, L_0 + C_0T]^2 \subset [0, R_0]^2$$

for all $t \in [0,T]$. The first inequality is due to the triangular inequality with the Wasserstein metric and our choice of T, and expresses the fact that σ_t^N is equibounded in $\mathcal{P}([0,R_0]^2)$. The second equation (4.2.12) proves (a) which ensures that σ_t^N is equilipschitz. Therefore, there exists a subsequence of $\{\sigma_t^N\}_N$ still denoted by $\{\sigma_t^N\}_N$ such that $\{\sigma_t^N\}_N$ converges narrowly to some σ_t for each t fixed independently of N.

We next show that σ_t solves (4.2.8) or equivalently (4.2.6) in view of remark 4.2.6 . For this purpose, we only have to show that up to some sequence $\left\{\mathbf{w}_t^N \sigma_t^N dt\right\}_N$ converges in the sense of distribution to $V_t[\sigma_t]\sigma_t dt$.

We recall that

$$V_{\tau[\frac{t}{\tau}]}[\sigma^N_{\tau[\frac{t}{\tau}]}] = \mathbf{w}^N_{\tau[\frac{t}{\tau}]} \tag{4.2.13}$$

Let $\phi \in C_c^1((0,T)\times\mathbb{R}^2,\mathbb{R}^2)$. Then, using (4.2.13), we have

$$\begin{aligned}
&\int_0^T dt \int_{\mathbb{R}^2} \langle\phi(t,x);\ \mathbf{w}_t^N\rangle d\sigma_t^N - \int_0^T dt \int_{\mathbb{R}^2} \langle\phi(t,x);\ V_t[\sigma_t]\rangle d\sigma_t = \\
&\left(\int_0^T dt \int_{\mathbb{R}^2} \langle\phi(t,x);\ \mathbf{w}_t^N\rangle d\sigma_t^N - \int_0^T dt \int_{\mathbb{R}^2} \langle\phi(t,x);\ \mathbf{w}^N_{\tau[\frac{t}{\tau}]}\rangle d\sigma^N_{\tau[\frac{t}{\tau}]}\right) \\
&+ \left(\int_0^T dt \int_{\mathbb{R}^2} \langle\phi(t,x);\ V_{\tau[\frac{t}{\tau}]}[\sigma^N_{\tau[\frac{t}{\tau}]}]\rangle d\sigma^N_{\tau[\frac{t}{\tau}]} - \int_0^T dt \int_{\mathbb{R}^2} \phi(t,x);\ V_t[\sigma^N_{\tau[\frac{t}{\tau}]}] d\sigma^N_{\tau[\frac{t}{\tau}]}\right) \\
&+ \left(\int_0^T dt \int_{\mathbb{R}^2} \langle\phi(t,x);\ V_t[\sigma^N_{\tau[\frac{t}{\tau}]}]\rangle d\sigma^N_{\tau[\frac{t}{\tau}]} - \int_0^T dt \int_{\mathbb{R}^2} \langle\phi(t,x);\ V_t[\sigma_t]\rangle d\sigma_t\right)
\end{aligned} \tag{4.2.14}$$

First, we look at the first term in the right handside of the equality in (4.2.14). We use (4.2.9) to obtain

$$\begin{aligned}
&\left|\int_0^T dt \int_{\mathbb{R}^2} \langle\phi(t,x);\ \mathbf{w}_t^N\rangle d\sigma_t^N - \int_0^T dt \int_{\mathbb{R}^2} \langle\phi(t,x);\ \mathbf{w}^N_{\tau[\frac{t}{\tau}]}\rangle d\sigma^N_{\tau[\frac{t}{\tau}]}\right| \\
&\le \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} dt \int_{\mathbb{R}^2} \left|\langle\phi\left(t, x + (t-\tau[\frac{t}{\tau}])\right) - \phi(t,x);\ \mathbf{w}^N_{\tau[\frac{t}{\tau}]}\rangle\right| d\sigma^N_{\tau[\frac{t}{\tau}]} \\
&\le \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} dt \int_{\mathbb{R}^2} \left|\phi\left(t, x + (t-\tau[\frac{t}{\tau}])\mathbf{w}^N_{\tau[\frac{t}{\tau}]}\right) - \phi(t,x)\right| |\mathbf{w}^N_{\tau[\frac{t}{\tau}]}| d\sigma^N_{\tau[\frac{t}{\tau}]}
\end{aligned} \tag{4.2.15}$$

We use the facts that $\sigma^N_{\tau[\frac{t}{\tau}]}$ has its support in $[0,R_0]^2$, ϕ is Lipschitz on $[0,T]\times[0,R_0]^2$ and the second equation of (4.2.12) to get

$$\begin{aligned}
&\sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} dt \int_{\mathbb{R}^2} \left|\phi\left(t, x+(t-\tau[\frac{t}{\tau}])\mathbf{w}^N_{\tau[\frac{t}{\tau}]}\right)-\phi(t,x)\right| |\mathbf{w}^N_{\tau[\frac{t}{\tau}]}| d\sigma^N_{\tau[\frac{t}{\tau}]} \\
&\leq \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} dt \int_{[0,R_0]^2} Lip(\phi) \left|t-\tau[\frac{t}{\tau}]\right| |\mathbf{w}^N_{\tau[\frac{t}{\tau}]}|^2 d\sigma^N_{\tau[\frac{t}{\tau}]} \\
&\leq C_0^2 Lip(\phi) \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} (k\tau - t) dt \\
&= C_0^2 Lip(\phi) \sum_{k=1}^N \int_0^\tau t dt \\
&= N C_0^2 Lip(\phi) \frac{\tau^2}{2} = C_0^2 Lip(\phi) \frac{T^2}{2N}
\end{aligned} \tag{4.2.16}$$

We combine (4.2.15) and (4.2.16) to obtain

$$\begin{aligned}
&\limsup_{N\to\infty} \left| \int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t,x);\ \mathbf{w}^N_t\rangle d\sigma^N_t - \int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t,x);\ \mathbf{w}^N_{\tau[\frac{t}{\tau}]}\rangle d\sigma^N_{\tau[\frac{t}{\tau}]} \right| \\
&\qquad \leq C_0^2 Lip(\phi) \limsup_{N\to\infty} \frac{T^2}{2N} = 0
\end{aligned} \tag{4.2.17}$$

To control the second term in (4.2.14), we use (4.2.4) to obtain

$$\begin{aligned}
&\left| \int_0^T dt \int \langle \phi(t,x);\ V_{\tau[\frac{t}{\tau}]}[\sigma^N_{\tau[\frac{t}{\tau}]}] - V_t[\sigma^N_{\tau[\frac{t}{\tau}]}]\rangle d\sigma^N_{\tau[\frac{t}{\tau}]} \right| \\
&\qquad \leq \|\phi\|_\infty \int_0^T \sup_{p\in\bar{\Delta}_{r_0}} \left| \mathbb{F}_{\tau[\frac{t}{\tau}]}(p) - \mathbb{F}_t(p) \right| dt
\end{aligned} \tag{4.2.18}$$

As $|t-\tau\left[\frac{t}{\tau}\right]| \leq \tau = \frac{T}{N}$ and $\mathbb{F}$ is continuous and bounded on $[0,T]\times\Delta_{r_0}$, we use the Lebesgue dominated convergence theorem to obtain that

$$\begin{aligned}
&\limsup_{N\to\infty} \left| \int_0^T dt \int \langle \phi(t,x);\ V_{\tau[\frac{t}{\tau}]}[\sigma^N_{\tau[\frac{t}{\tau}]}] - V_t[\sigma^N_{\tau[\frac{t}{\tau}]}]\rangle d\sigma^N_{\tau[\frac{t}{\tau}]} \right| \\
&\qquad \leq \|\phi\|_\infty \limsup_{N\to\infty} \int_0^T \sup_{p\in\Delta_{r_0}} \left| \mathbb{F}_{\tau[\frac{t}{\tau}]}(p) - \mathbb{F}_t(p) \right| dt = 0
\end{aligned} \tag{4.2.19}$$

Let's work on the third term in (4.2.14).

We note that

$$\left| \int_{\mathbb{R}^2} \langle \phi(t,x), V_t[\sigma^N_{\tau[\frac{t}{\tau}]}]\rangle d\sigma^N_{\tau[\frac{t}{\tau}]} \right| \leq C_0 \|\phi\|_\infty \tag{4.2.20}$$

Using (a), we have

$$W_1\left(\sigma_{\tau\left[\frac{t}{\tau}\right]}, \sigma_t\right) \le C_0 \left| t - \tau\left[\frac{t}{\tau}\right]\right| \le \frac{C_0 T}{N}$$

And so, As N goes to ∞, $\sigma_{\tau\left[\frac{t}{\tau}\right]}$ converges narrowly to σ_t and lemma 4.2.3 ensures that

$$\int_{\mathbb{R}^2} \langle \phi(t,x);\ V_t[\sigma^N_{\tau\left[\frac{t}{\tau}\right]}]\rangle d\sigma^N_{\tau\left[\frac{t}{\tau}\right]} \text{ converges a.e to } \int_{\mathbb{R}^2} \langle \phi(t,x);\ V_t[\sigma_t]\rangle d\sigma_t \tag{4.2.21}$$

We combine (4.2.20) and (4.2.21) and use the Lebesgue dominated convergence theorem to obtain that

$$\int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t,x) V_t[\sigma^N_{\tau\left[\frac{t}{\tau}\right]}]\rangle d\sigma^N_{\tau\left[\frac{t}{\tau}\right]} \text{ converges to } \int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t,x), V_t[\sigma_t]\rangle d\sigma_t \tag{4.2.22}$$

In view of (4.2.14), (4.2.17) (4.2.20) and (4.2.22) we have

$$\limsup_{N\to\infty} \left| \int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t,x), \mathbf{w}^N_t \rangle d\sigma^N_t - \int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t,x), V_t[\sigma_t]\rangle d\sigma_t \right| = 0$$

As ϕ is arbitrary, we obtain that $\left\{\mathbf{w}^N_t \sigma^N_t dt\right\}_N$ converges in the sense of distribution to $V_t[\sigma_t]\sigma_t dt$ which concludes the proof. $\square$

APPENDIX A

MONOTONE REARRANGEMENT AND PROPRIETIES.

We use the following lemmas in the sequel.

Lemma A.0.9 *(Approximation by convolution) Let $\eta \in \mathcal{P}_p(\mathbb{R})$ and let ρ_ε be a family of nonnegative mollifiers such that*

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon}), \qquad \int_{\mathbb{R}} \rho(x)dx = 1, \qquad m = \int_{\mathbb{R}} |x|^p \rho(x)dx < \infty$$

Then if $\eta_\varepsilon = \rho_\varepsilon * \eta$

$$W_p(\rho_\varepsilon, \rho) \leq m\varepsilon$$

Therefore ρ_ε converges narrowly to ρ.

Proof: See [[1] Page 156]

Remark A.0.10 *Note that if $\rho > 0$ in lemma A.0.9 then $\eta_\varepsilon > 0$.*

Theorem A.0.11 *(Brenier's theorem) Let μ and ν be probability measures with finite second moments. If μ is absolutely continuous with respect to Lebesgue, then there exists a borel function T_0 such that*

$$T_0 = \nabla\varphi \quad \mu - a.e \qquad (\varphi \textit{ convex}) \qquad \textit{and} \qquad T_{0\#}\mu = \nu.$$

Moreover T_0 is the unique minimizer of

$$\inf_{T_{0\#}\mu=\nu} \int_{\mathbb{R}^2} |x - Tx|^2 d\mu(x)$$

Proof: See [[46], Page 67]

Lemma A.0.12 *On the real line, the set of all the functions which are gradients of convex functions is equal to the class of functions which coincide with monotone non decreasing functions almost everywhere.*

Proof: 1. we recall that if φ is convex, φ'_+ and φ'_- (respectively, right and left derivative φ) exist are finite at each point and are non-decreasing and so φ is differentiable except at countably many points at most and φ' coincides with a monotone non-decreasing function a.e.

2. Conversely, assume f is monotone non-increasing. Then f is borel measurable and locally integrable. Set

$$\varphi(x) = \int_0^x f(u)du$$

Note that φ is continuous. We next show that φ is convex.

For any $a, b \in \mathbb{R}$, as f is non decreasing,

$$f_{|[a,\frac{a+b}{2}]} \leq f\left(\frac{a+b}{2}\right) \leq f_{|[\frac{a+b}{2},b]}$$

Thus, we have

$$\int_a^{\frac{a+b}{2}} f(z)dz \leq \int_a^{\frac{a+b}{2}} f\left(\frac{a+b}{2}\right) dz = \int_{\frac{a+b}{2}}^b f\left(\frac{a+b}{2}\right) dz \leq \int_{\frac{a+b}{2}}^b f(z)dz$$

and so,

$$\int_a^{\frac{a+b}{2}} f(z)dz \leq \int_{\frac{a+b}{2}}^b f(z)dz$$

that is,

$$\int_0^{\frac{a+b}{2}} f(z)dz - \int_0^a f(z)dz \leq \int_0^b f(z)dz - \int_0^{\frac{a+b}{2}} f(z)dz$$

This, we rewrite as

$$2\int_0^{\frac{a+b}{2}} f(z)dz \leq \int_0^b f(z)dz + \int_0^a f(z)dz.$$

Thus,

$$\varphi(\frac{a+b}{2}) \leq \frac{\varphi(a)+\varphi(b)}{2}$$

We conclude that φ is convex. As f is locally integrable, the Lebesgue differentiation theorem ensures that $\varphi' = f$ *a.e.* $\square$

Definition A.0.13 *A function $h : [0, H] \longrightarrow \mathbb{R}$ is said to be a rearrangement of a function g if for every measurable function F such that $F \circ g \in L^1[0, H]$, we have $F \circ h \in L^1[0, H]$ and*

$$\int_0^H F \circ g \, dz = \int_0^H F \circ h \, dz.$$

In other words,

$$g_{\#}\chi_{[0,H]} = h_{\#}\chi_{[0,H]}.$$

Lemma A.0.14 *Let $h,\ g : [0, H] \longrightarrow \mathbb{R}$ be L^2 functions. Then*

(i) There exists a unique (up to a set of Lebesgue measure zero) nondecreasing rearrangement $h^{\#}$ of h.

(ii)

$$||h^{\#} - g^{\#}||_{L^2[0,H]} \leq ||h - g||_{L^2[0,H]}$$

(iii) $(N \circ h)^{\#} = N \circ h^{\#}$ if N is monotone non decreasing.

Proof: 1. Set

$$\eta_0 = \frac{1}{H}\chi_{[0,H]}\mathcal{L}^1 \qquad \eta = h_{\#}\eta_0$$

As $h \in L^2[0, H]$, we have that η is of finite second moment. We use theorem A.0.11 to obtain the existence of a unique monotone nondecreasing function $h^{\#}$ such $h^{\#}_{\#}\eta_0 = \eta$, which completes the proof of (i).

2. Set

$$\bar{\eta} = g_{\#}\eta_0 \qquad \gamma_0 = (h \times g)_{\#}\, \eta_0$$

Note that $\bar{\eta} \in \mathcal{P}_2(\mathbb{R})$, $\gamma_0 \in \Gamma(\eta, \bar{\eta})$ and

$$W_2^2(\eta, \bar{\eta}) \leq \int_{[0,H]^2} |x - y|^2 d\gamma_0 = ||h - g||^2_{L^2[0,H]}$$

To obtain (ii), we are to show that

$$W_2(\eta, \bar{\eta}) = ||h^{\#} - g^{\#}||_{L^2[0,H]}. \tag{A.0.23}$$

As $\eta \in \mathcal{P}_2(\mathbb{R})$, lemma A.0.11 provides a family of probability measures $\{\eta^n\}_{n=1}^{\infty}$ absolutely continuous with respect to Lebesgue such that $W_2^2(\eta^n, \eta) \leq \frac{1}{n}$. We may assume $\eta^n > 0$ in light of remark A.0.10. For each $n \in \mathbb{N}$, we define

$$M_n(x) := \eta_n(-\infty, x), \qquad h_n^*(z) := \sup\{x \in \mathbb{R} : HM_n(x) \leq z\}$$

is absolutely continuous with respect to Lebesgue and strictly positive, M_n is continuous and strictly increasing. Therefore, $M_n : \mathbb{R} \longmapsto (0, 1]$ is bijective.

Thus,

$$h_n^*(z) = \sup\left\{x \in \mathbb{R} : x \leq M_n^{-1}(z/H)\right\} = M_n^{-1}(z/H.)$$

and so, h_n^* is invertible with inverse $\lambda \circ M_n$ (here, $\lambda(x) = Hx$).

Fix $x \in \mathbb{R}$,

$$\begin{aligned}
h_{n\#}^*\eta_0\left((-\infty, x]\right) &= \frac{1}{H}\mathcal{L}^1_{|_{[0,H]}}\left(\{z \in [0, H] : h_n^*(z) \leq x\}\right) \\
&= \frac{1}{H}\mathcal{L}^1_{|_{[0,H]}}\left(\left\{z \in [0, H] : M_n^{-1}(z/H) \leq x\right\}\right) \\
&= \frac{1}{H}\mathcal{L}^1_{|_{[0,H]}}\left(\{z \in [0, H] : z \leq HM_n(x)\}\right) \\
&= \frac{1}{H}\mathcal{L}^1_{|_{[0,H]}}\left(H\{z \in [0, 1] : z \leq M_n(x)\}\right) \\
&= \mathcal{L}^1_{|_{[0,H]}}\left(0, M_n(x)\right) = M_n(x) = \eta^n(\infty, x)
\end{aligned}$$

We have used the scaling property of the Lebesgue measure is the fifth equation. Therefore,

$$h_{n\#}^*\eta_0 = \eta^n. \tag{A.0.24}$$

This implies that

$$\lambda \circ M_{n\#}\eta^n = \eta_0.$$

Now, as M_n is strictly increasing, h_n^* is strictly increasing and by using the theorem of Helly, we may assume that h_n^* converges to h^* pointwise for some h^* monotone increasing. Let F be a continuous and bounded function of $\mathbb{R}$. Then, in view of

(A.0.24), we have

$$\int_0^H F \circ h_n^* d\eta_0 = \int_0^H F d\eta^n$$

As $n \to \infty$, by using the theorem of dominated convergence on the left hand side and the narrow convergence on the right hand side of the previous equality we obtain :

$$\frac{1}{H}\int_0^H F \circ h^* dz = \int_0^H F \circ h^* d\eta_0 = \int_0^H F d\eta = \frac{1}{H}\int_0^H F \circ h dz \tag{A.0.25}$$

By a density argument, (A.0.25) holds for all $F \in L^1(\eta)$. Thus, by the uniqueness of the monotone rearrangement

$$h^{\#} = h^* \quad a.e$$

As $h^{\#} \circ \lambda \circ M_n$ is monotone non-decreasing and $h^{\#} \circ \lambda \circ M_{n\#}\eta^n = \eta$, theorem A.0.9 ensures that $h^{\#} \circ \lambda \circ M_n$ minimizes

$$W_2^2(\eta^n, \eta) = \inf_T \left\{ \int_{[0,H]} |x - Tx|^2 d\eta : T_{\#}\eta^n = \eta \right\}$$

and so

$$\begin{aligned} ||h_n^* - h^{\#}||^2_{L^2(\eta_0)} &= \int_{[0,H]} |h_n^*(y) - h^{\#}(y))|^2 d\eta_0 \\ &= \int_{[0,H]} |h_n^* \circ \lambda \circ M_n(x) - h^{\#} \circ \lambda \circ M_n(x)|^2 d\eta^n \\ &= \int_{[0,H]} |x - h^{\#} \circ \lambda \circ M_n(x)|^2 d\eta^n \\ &= W_2^2(\eta^n, \eta) \end{aligned} \tag{A.0.26}$$

A similar argument shows that

$$||h_n^* - g^{\#}||^2_{L^2(\eta_0)} = W_2^2(\eta^n, \bar{\eta}) \tag{A.0.27}$$

We combine (A.0.26) and (A.0.27) and use the triangular inequality to obtain

$$\begin{aligned} ||h_n^* - h^{\#}||_{L^2(\eta_0)} + |h_n^* - g^{\#}||_{L^2(\eta_0)} &\le W_2(\eta^n, \eta) + W_2(\eta^n, \bar{\eta}) \\ &\le 2W_2(\eta^n, \eta) + W_2(\eta, \bar{\eta}) \end{aligned} \tag{A.0.28}$$

As $W_2(\eta^n, \eta) \leq \frac{1}{n}$, (A.0.28) implies that

$$||g^{\#} - h^{\#}||_{L^2(\eta_0)} \leq \frac{2}{n} + W_2(\eta, \bar{\eta}) \tag{A.0.29}$$

We easily check that

$$W_2(\eta, \bar{\eta}) \leq ||g^{\#} - h^{\#}||_{L^2(\eta_0)} \tag{A.0.30}$$

We combine (A.0.29) and (A.0.30) to obtain (A.0.23).

4. Assume N is monotone. We easily check that $N \circ h^{\#}$ is a rearrangement of $N \circ h$. As $h^{\#}$ is monotone, $N \circ h^{\#}$ is monotone non increasing and the uniqueness in (i) ensures that $(N \circ h)^{\#} = N \circ h^{\#}$. $\square$

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